An Elementary Introduction to

Lie Algebras for Physicists

Douglas W. McKenzie

A primer to prepare a prospective student, whose interest is primarily physics, for the Lie group / algebra portion of a group theory course

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1. Introduction

The purpose of this review is to present a practical, relatively easy to follow description of the basics of Lie groups and algebras focusing on the material of interest to physicists. It is an 'introduction' because it only discusses some of the most important points and leaves much out. It is 'elementary' because it does not leave much to the imagination of the student, rather it is very explicit. The motivation for the review is the difficulty that the author had in learning these basics himself. Hopefully, after reading the review, the student can approach other texts with a level of confidence when going into the subject more deeply.

The idea that it might even be possible for Lie groups and algebras to be presented in an accessible way to the first time student derived from Bob Klauber's *Student Friendly Quantum Field Theory*. I found this text to be, hands down, the most accessible text for first approaching QFT. Following one of the philosophies of that book, every effort has been made to avoid brevity and conciseness which are truly obstacles for the first time student. I would also like to thank Pietro Longhi who peeled back some of the layers of this onion-like material for me. My thanks also to Patrick Cooper who helped me understand what a typical physics student needs to understand about Lie algebras.

Since the purpose of this review is to be clear and accessible, it would be very helpful to receive comments and suggestions particularly on how it can be made clearer and more accessible. Reporting any errors would also be enormously appreciated. My email is

mckenziedouglas@gmail.com

or you can use the Contact Me tab on the website

http://www.liealgebrasintro.com.

We assume that the reader has a working knowledge of finite group theory, linear algebra and quantum mechanics.

2. Overview

The areas of application of Lie Groups to physics range from understanding how angular momentum in quantum mechanics behaves, to describing families of elementary particles and their behavior, to succinctly expressing the symmetries and invariances behind gauge theories. This review goes into some detail regarding angular momentum. It gives a simple example of families of elementary particles. Other than those physical applications it simply provides a foundation for further study.

The first topic introduces the basic calculation tools that will be used - The generators of a Lie group and the use of exponentiation of those generators to recover the original group elements. SO(2), a one parameter rotation group, is used as the example. We learn here that the generators of a Lie algebra contain all the information about the group.

The second topic introduces Lie algebras by applying the tools from the first topic to a group with 3 parameters (hence, three generators) called SO(3). The Lie algebra is defined by the commutation relationships of these generators. Then another group, SU(2) which is locally isomorphic to SO(3). is introduced because the SU(N) groups will be our primary groups of interest.

The third topic begins with the development of raising and lowering operators – basic tools for studying the states of a Lie algebra. The states we are interested in are the angular momentum states of a multi-particle system. The raising and lowering operators are then applied to the tensor product of the individual particle representations to extract the irreducible representations of the multi-particle states. Lastly, we look at isospin states in a manner completely analogous to angular momentum with the difference being that the states become particles!

The fourth topic explores another aspect of Lie algebras known as roots. These are eigenvalues which like the generators contain all the information about the group in a remarkably concise manner. With these roots we can find irreducible representations relatively easily. They also permit a powerful classification of high energy particles as well as of the Lie algebras themselves. This topic may not be of interest if the reader's course of study does not involve this deeper look into Lie algebras.

3. What is a Lie Group? What is a Lie Algebra?

3.1. What is a Lie Group

A Lie group is a group whose group elements are specified by one or more continuous parameters which vary smoothly. We consider a simple example, the SO(2) group – rotation in two dimensions. The group is characterized by a single parameter, θ, the angle of rotation. The elements of this group are the individual rotation operators, for example, R(45^o) indicating a rotation counter clockwise of 45^o, is an element of the group. The group operation is the consecutive application of two rotation operators. For example, R(15^o) R(30^o) = R(45^o) indicates that a 30^o rotation followed by a 15^o rotation produces the same result as a 45^o rotation. These group elements are often represented as 2 dimensional matrices of the form

$$M(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \quad (3.1.1)$$

These matrices rotate any point on the plane around the origin counter-clockwise through the angle θ. For example, M(45^o) rotates the point (0,1) to the point $(-\sqrt{1/2}, \sqrt{1/2})$. When we find a representation, such as this one, that explicitly shows the nature of the group elements, we will refer to it as a **natural representation**.[1]

This group is referred to as SO(2) - S for Special i.e. the determinant is 1, O for Orthogonal i.e. the elements of the matrices are real and the transpose of the matrix is its inverse and 2 for the

[1] The term 'natural representation' is not used consistently in the literature.

dimension of the space in which the rotation takes place.

Notice that the actual elements of the group, the rotation operators, are quite abstract and they can be described in words but it would be difficult to perform meaningful calculations with these elements as given. The idea of a representation is to make the group elements susceptible to rigorous analyses. Here, and usually for physicists, group elements will be represented by matrices and our focus will be on those matrices (except where noted). Hence, we will use the notation of M to label the representation of group element R. In other texts, D or Γ are used but we use the label M to make it clear that we are using matrices.

The key property of these matrices is that they must reflect the group structure i.e. since

$$R(15^o)\, R(30^o) = R(45^o)$$

then the representatives of those operations, $M(15^o)$, $M(30^o)$ and $M(45^o)$, must satisfy

$$M(15^o)M(30^o) = M(45^o).$$

We demonstrate this by performing the matrix multiplication and simplifying

$$\begin{pmatrix} cos(15^o) & -sin(15^o) \\ sin(15^o) & cos(15^o) \end{pmatrix}\begin{pmatrix} cos(30^o) & -sin(30^o) \\ sin(30^o) & cos(30^o) \end{pmatrix}$$

$$= \begin{pmatrix} cos(15^o + 30^0) & -sin(15^o + 30^0) \\ sin(15^o + 30^0) & cos(15^o + 30^0) \end{pmatrix}$$

since

$$\cos(A+B) = cos(A)cos(B) - sin(A)sin(B) \text{ and}$$
$$\sin(A+B) = sin(A)cos(B) + cos(A)sin(B).$$

There are other matrix representations of this group. Some have the same dimensionality, i.e. 2 of the example above. Typically, representations having the same dimensionality are **equivalent** to one another in that their matrices have a 1 to 1 relationship[2] with each other and each pair of matrices are related through the same similarity transformation, S, i.e. $M_2(p_2) = S^{-1}M_1(p_1)S$. We use p_1 *and* p_2 to show that the two representations may have different parameterizations. Because of this equivalence we will not consider any but one representation of a particular dimensionality unless we are making a specific point.[3] Other representations will have different dimensionalities and these can have important physical consequences.

3.2. What is a Lie Algebra?

We have seen above, equation (3.1.1), how the SO(2) group can be described using a matrix

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[2] We use the term '1 to 1' instead of 'bijective' or '1 to 1 onto' because it is more evocative. Any loss in rigor should not cause any problems.

[3] There are some situations where two inequivalent representations of the same dimension exist and have physical consequences but they are beyond the scope of this review.

representation of the form

$$M(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \tag{3.2.1}$$

We will see shortly that there is an even simpler way of characterizing the SO(2) group which does not even involve the parameter θ! The result, in this case, is a matrix

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{3.2.2}$$

which stores in itself all of the information about the group. This matrix is called a **generator** of the SO(2) **Lie algebra**. The Lie algebra is a mathematical structure that underlies the group structure. It is a linear algebra and it is not a group itself. A key property is that its bilinear operation is a commutator which we will discuss later. In section 8.1 a formal definition of Lie algebras is given.

In the next section, we will define and derive the SO(2) generator and show that it stores all of the information about the group. In later sections, we will gradually develop an introductory understanding of Lie algebras

4. A Trivial Lie Algebra – The Basic Tools

In this section we introduce infinitesimal generators (aka generators) as a compact representation of a Lie algebra. We will use the natural representation for SO(2) discussed above in section 3.1 which reflects the rotational nature of the group. We'll use that representation to define and calculate the generator of the Lie algebra.

We'll exponentiate the generator and show that all group elements can be re-derived from the exponentiated form. Thus, the generator contains all of the group structure information.

4.1. Calculating the Generators

Finding the generators of the Lie algebra from a representation of the Lie group is the first step. We begin by considering only group elements in the vicinity of the identity element, $M(0)$. We can expand $M(\theta)$ as a Taylor expansion around θ=0. Thus,

$$M(\theta) = I + \theta M'(0) + \frac{\theta^2 M''(0)}{2!} + \frac{\theta^3 M'''(0)}{3!} \cdots$$

For very small $\theta, \varepsilon,$,

$$M(\varepsilon) \approx I + \varepsilon M'(0).$$

This can also be written as

$$M(\varepsilon) \approx I + i\varepsilon X$$

where

$$X = -iM'(0)^{4} \tag{4.1.1}$$

X is referred to as the generator for this Lie group. More accurately, it is a representation of the generator of this Lie group. X is a matrix but we will continue to call it X, rather than M, because of its special status as a representation of the generator rather than a representation of the group elements. We will see later that these two behave differently and it is important to keep track of with which one we are working.

There is only one generator for SO(2) because there is only one parameter and this is why we refer to it as a trivial Lie algebra. There need to be at least three generators for a non-trivial Lie algebra. The advantage of using SO(2) is that various calculation procedures are clearer for this simple group. Using the representation of the group elements that we identified above, equation (3.2.1), we can calculate X

$$X = -iM'(0) = -i\frac{d}{d\theta}\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}_{\theta=0}$$

$$= -i\begin{pmatrix} -sin(\theta) & -cos(\theta) \\ cos(\theta) & -sin(\theta) \end{pmatrix}_{\theta=0} = -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{4.1.2}$$

The same result as equation (3.2.2). In the next section, we show that with this single matrix we can generate a representation of the group demonstrating that the generator contains all the information of the group structure

4.2. Recover the Group Elements with Exponentiation

Now we can use the generator, X, to generate (hence the name) a representation, $M^{generated}(\theta)$, for finite θ's. In this sense, we are not really recovering the group elements rather we are recovering a representation of the group elements. We take the limit as $N \to \infty$ of $(I + i\frac{\theta}{N}X)^N$ and get $e^{i\theta X}$. We show below that $e^{i\theta X} = M^{generated}(\theta)$ thus demonstrating the recovery of a representation of the group elements.

We expand the exponential in a Taylor expansion with all terms.

$$e^{i\theta X} = I + (i\theta X) + \frac{(i\theta X)^2}{2!} / + \frac{(i\theta X)^3}{3!} + \frac{(i\theta X)^4}{4!} + \cdots \tag{4.2.1}$$

This can be simplified by noting that even powers of X are I and odd powers of X are X. Then, grouping the even powers of θ together and the odd powers together we get

[4]Some texts define X as $+iM'(\theta)$ and others as just $M'(\theta)$. The i (either $+$ or $-$) ensures that if the $M's$ are unitary $(M^{\dagger} = M^{-1})$, then the X's will be Hermitian $(X^{\dagger} = X)$.

$$= I + \frac{(i\theta)^2 I}{2!} + \frac{(i\theta)^4 I}{4!} + \cdots + (i\theta)X + \frac{(i\theta)^3 X}{3!} + \cdots$$

$$= I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) + \cdots + iX\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= I\,cos(\theta) + i\,X\,sin(\theta)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} cos(\theta) + i \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} sin(\theta) \tag{4.2.2}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} cos(\theta) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} sin(\theta)$$

$$= \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} = M^{generated}(\theta) \tag{4.2.3}$$

We have identically recovered the original representation. Using this exponentiation procedure we do not always recover the original representation. In the next section we will use two different original representations and we will see what the generators and the generated representations are.

4.3. Starting With a Different Representation

We started with a representation which clearly reflects the rotational nature of the group and saw that the generator maintained that information by recovering the identical representation. Now we want to show that even if we start with a representation that does not reflect the rotational nature of the group the generator will recover that information regardless. This is the one time that we will discuss multiple representations of the same dimension. We start with another matrix which has determinant = 1 and the transpose is the inverse:

$$M_2(x) = \begin{pmatrix} \sqrt{1-x^2} & -x \\ x & \sqrt{1-x^2} \end{pmatrix} \tag{4.3.1}$$

So this matrix serves as another representation of the SO(2) group. We calculate the generator, X_2 , as

$$X_2 = -iM_2'(x)_{x=0} = -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

which is the same as the generator calculated from the natural representation, equation (4.1.2). Since the generator is the same as before, the form of $M_2^{generated}(\theta)$ that is derived using exponentiation is also the same, i.e.

$$M_2^{generated}(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

Note that this representation is not the original one, $M_2(x)$ (equation (4.3.1)), that we used to derive the generator. It is interesting to note that if we used another representation, $M_3(x)$, the transpose of $M_2(x)$, i.e.

$$M_3(x) = M_2^T(x) = \begin{pmatrix} \sqrt{1-x^2} & x \\ -x & \sqrt{1-x^2} \end{pmatrix} \tag{4.3.2}$$

as our original representation then we would generate

$$M_3^{generated} = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$$

which can be viewed either as the transpose of the first original representation or as the first original with $-\theta$ substituted for θ. If two original representations are equivalent then the two generated representations will also be equivalent. The three original representations that we have considered, $M_1,\ M_2\ and\ M_3$ (equations (3.2.1), (4.3.1), and (4.3.2)), are equivalent if $0 \le \theta < 2\pi\ and\ 0 \le x \le 1$. This is the first of two places where we consider several representations of the same dimension.

4.4. Summary of Main Points

A generator of the algebra of a one parameter Lie group, SO(2), was derived. The generator was derived from a particular representation of the group which expressed the group structure naturally,

Exponentiating the generator was shown to recreate the starting representation of the group which demonstrates that the generator contains all of the information of the group structure.

Other starting representations were used and the resulting generators, when exponentiated, yielding representations which were not the same as the starting representation.

5. A More Instructive Lie Algebra – The Structure Constants[5]

In this section we are going to develop the general Lie algebra which means for a Lie group with an arbitrary number of parameters. We will apply that general formulation to a three parameter group, SO(3). Three is the smallest number of parameters that requires the general formulation.

5.1. SO(3) - A Lie Group with Three Parameters

[5] This would have been called ‘A Simple Lie Algebra’ except that ‘simple Lie algebra’ is an important type of Lie algebra and all we wish to convey is that this Lie algebra is more complex than the trivial one but also rather simple.

The group of rotations in three dimensions, SO(3), is a Lie group with 3 parameters which are the 3 angles of rotation, θ_1, θ_2 *and* θ_3 around the three Cartesian coordinate axes. There are other sets of three angles that we could use but this one is easy to follow. As with SO(2), we first describe a natural representation. The building blocks of the representation that we will use are

$$M_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_1) & -sin(\theta_1) \\ 0 & sin(\theta_1) & cos(\theta_1) \end{pmatrix}$$

$$M_2(\theta_2) = \begin{pmatrix} cos(\theta_2) & 0 & -sin(\theta_2) \\ 0 & 1 & 0 \\ sin(\theta_2) & 0 & cos(\theta_2) \end{pmatrix} \tag{5.1.1}$$

$$M_3(\theta_3) = \begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There are a number of ways these three matrices can be combined to form a representation of SO(3). We will use $M(\theta_1, \theta_2, \theta_3) = M_1(\theta_1) M_2(\theta_2) M_3(\theta_3)$.

5.2. Calculating the Generators

The general formula for calculating the generators, regardless of how many parameters there are, is

$$X_i = -i \frac{\partial M(\vec{\theta})}{\partial \theta_i} \bigg|_{\vec{\theta}=0} \tag{5.2.1}$$

which is the multi-variate form of equation (4.1.1). For SO(3), we will calculate three generators. The generators corresponding to the natural representation above, equation (5.1.1) are shown below.

$$X_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = -i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.2}$$

The Lie algebra is a vector space within which the generators are operators. The dimension, d, of that space is the number of generators. So, in this case, $d = 3$. Because the space for SO(2) had $d = 1$, the algebra was trivial but we were able to make three observations that are true for all Lie algebras.

1. The representations of the generators are calculated by taking the derivative of the group

element representation at the identity element.
2. Exponentiating the generators creates another (possibly identical but not necessarily) group element representation.
3. A clear distinction needs to be made between the matrices which represent the generators and the matrices that represent the group elements – they belong to different spaces and they behave differently. And, in fact, the Lie algebra, is not even a group.

5.3. Exponentiation and Deriving the General Lie Algebra

When we exponentiated the generator from the single parameter representation we did not check to make sure that the generated representation was, in fact, really a representation of the group at all. We didn't need to check because it obviously was since the generated representation was identical with the original representation. In the multi-parameter case, it is not at all trivial to establish that the generated results are a representation of the group. We will continue with the program of exponentiation but now we will need to determine the conditions on the generators that must be met to ensure that the matrices generated by the exponentiation are a representation of the group. The exponentiation of the generators when there is more than one parameter takes the form below.

$$e^{i\vec{\theta}\cdot\vec{X}} \quad or \quad e^{i\theta_i X_i} \tag{5.3.1}$$

where the Einstein summation convention is used in the second expression. To ensure that the final result of exponentiation is a valid representation of the group, it is necessary that the generators reflect the group structure i.e. if group elements, A, B and C, satisfy

$$AB = C$$

then

$$e^{i\theta_{Ai}X_i}e^{i\theta_{Bj}X_j} = e^{i\theta_{Ci}X_i} \tag{5.3.2}$$

where the θ_{Ci}'s are to be determined. The group condition can only be met if the X_i's satisfy certain constraints that we will now derive. These constraints form the basis of the Lie algebra.

Using the Taylor expansion of equation (5.3.2), we get

$$\left\{I + (i\theta_{Ai}X_i) + \frac{(i\theta_{Ai}X_i)^2}{2}\cdots\right\}\left\{I + (i\theta_{Bj}X_j) + \frac{(i\theta_{Bj}X_j)^2}{2}\cdots\right\}$$
$$= I + (i\theta_{Ci}X_i) + \frac{(i\theta_{Ci}X_i)^2}{2}\cdots$$

Retaining only the terms up to second order we get

$$I + (i\theta_{Ai}X_i) + (i\theta_{Bj}X_j)$$
$$+\frac{(i\theta_{Ai}X_i)^2}{2} + \frac{(i\theta_{Bj}X_j)^2}{2} + (i\theta_{Ai}X_i)(i\theta_{Bj}X_j)$$

$$= I + (i\theta_{Ci} X_i) + \frac{(i\theta_{Ci} X_i)^2}{2}$$

We carefully rearrange these terms paying particular attention to the indices. We also approximate $\vec{\theta}_C$ with $\vec{\theta}_A + \vec{\theta}_B$ in the quadratic term of the RHS. This is the result of expanding $\vec{\theta}_C$ in a power series of $\vec{\theta}_A + \vec{\theta}_B$ and dropping the quadratic and higher terms.

$$I + (i\theta_{Ai} X_i) + (i\theta_{Bi} X_i)$$
$$- \frac{(\theta_{Ai} X_i)(\theta_{Aj} X_j)}{2} - \frac{(\theta_{Bi} X_i)(\theta_{Bj} X_j)}{2} - (\theta_{Ai} X_i)(\theta_{Bj} X_j)$$
$$= I + (i\theta_{Ci} X_i) - \frac{(\theta_{Ai} + \theta_{Bi}) X_i (\theta_{Aj} + \theta_{Bj}) X_j}{2}.$$

Gathering terms of $X_i \; and \; X_i X_j$ gives

$$I + i(\theta_{Ai} + \theta_{Bi}) X_i$$
$$- \left(\frac{\theta_{Ai}\theta_{Aj}}{2} + \frac{\theta_{Bi}\theta_{Bj}}{2} + \theta_{Ai}\theta_{Bj} \right) X_i X_j$$
$$= I + (i\theta_{Ci}) X_i - \left(\frac{\theta_{Ai}\theta_{Aj}}{2} + \frac{\theta_{Ai}\theta_{Bj}}{2} + \frac{\theta_{Bi}\theta_{Aj}}{2} + \frac{\theta_{Bi}\theta_{Bj}}{2} \right) X_i X_j.$$

Subtracting the LHS from both sides gives

$$0 = i(\theta_{Ci} - \theta_{Ai} - \theta_{Bi}) X_i + \frac{(\theta_{Ai}\theta_{Bj} - \theta_{Bi}\theta_{Aj})}{2} X_i X_j.$$

$$0 = i(\theta_{Ci} - \theta_{Ai} - \theta_{Bi}) X_i + \frac{\theta_{Ai}\theta_{Bj}}{2} X_i X_j - \frac{\theta_{Bi}\theta_{Aj}}{2} X_i X_j.$$

Switching i and j in the last term

$$0 = i(\theta_{Ci} - \theta_{Ai} - \theta_{Bi}) X_i + \frac{\theta_{Ai}\theta_{Bj}}{2} X_i X_j - \frac{\theta_{Bj}\theta_{Ai}}{2} X_j X_i.$$

Since $\theta_{Ai} \; and \; \theta_{Bj}$ are numbers, they commute so we can write

$$0 = i(\theta_{Ci} - \theta_{Ai} - \theta_{Bi}) X_i + \frac{\theta_{Ai}\theta_{Bj}}{2} (X_i X_j - X_j X_i).$$

Recognizing the commutator, changing the index on the linear term and moving the linear term to the other side we get

$$-i(\theta_{Ck}-\theta_{Ak}-\theta_{Bk})X_k=\frac{\theta_{Ai}\theta_{Bj}}{2}[X_i,X_j]$$

$\left(\theta_{Ck}-\theta_{Ak}-\theta_{Bk}\right)$, on the LHS is quadratic in $\left(\theta_{Ak},\,\theta_{Bk}\right)$ since θ_{Ck} was approximated with $\left(\theta_{Ak}+\theta_{Bk}\right)$ with quadratic terms being ignored. Thus,

$$i\,Quadratic(\theta_{Ak},\,\theta_{Bk})X_k=Quadratic(\theta_{Ai},\,\theta_{Bj})[X_i,X_j]$$

The LHS is a linear combination of $X's$'s so the RHS must be also, hence,

$$[X_i,X_j]=iC_{ij}^kX_k \tag{5.3.3}$$

This relationship is the defining relationship for the Lie algebra and arises from the requirement that the exponentiated form of the generators satisfies the group multiplication rule. The C_{ij}^k's are called the **structure constants** of the Lie algebra. The factor i is inserted so that all of the C_{ij}^k's are real. Note that we never actually needed to solve for $\vec{\theta}_C$.

If we had done a more complex calculation using higher order terms we still would have obtained equation (5.3.3) as the relationship that ensures that the group multiplication rule is maintained. Equally amazing is that these structure constants apply to all representations of the group. This is our first example of a property of the Lie algebra which is true for all representations.

Usually the derivation of the structure constants can be quite complicated but since we are dealing with SO(3) we can calculate the C_{ij}^k's easily since the 'sum over k' just gives the 'other' generator on the RHS of equation (5.3.3).

5.4. Showing that the SO(3) Generators Satisfy the General Lie Algebra

The C_{ij}^k's for SO(3) can be determined by first calculating $[X_i,\,X_j]$ and then comparing the result to iX_k. They should be proportional and C_{ij}^k is equal to the proportionality constant. We will perform these calculations for. $X_1,\ X_2$ *and* X_3 come from equation (5.2.2).

$$[X_1,X_2]=(-i)\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}(-i)\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}-(-i)\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}(-i)\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

$$=(-1)\begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\end{pmatrix}-(-1)\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}=iC_{12}^3X_3$$

with $C_{12}^{3} = +1$. If the other two, positive elements of C_{ij}^{k} are calculated, we will see that the SO(3) generators satisfy the relationship

$$[X_i, X_j] = i\varepsilon_{ijk} X_k \tag{5.4.1}$$

where ε_{ijk} is the Levi-Civita tensor of three dimensions. This is the bilinear operation for the SO(3) Lie algebra.

5.5. Introducing the SU(2) Group

We could use the SO(3) group to analyze the angular momentum of multi-particle systems in three space but there is another group, SU(2), which is locally isomorphic to SO(3), that we will use instead. SU(2) is not only important for studying rotations but it is also important when studying higher dimensional Lie algebras. We said before that we would use the Lie algebra for SO(3) to analyze angular momentum of multi-particle systems. Now we are saying that we are going to use SU(2) rather than SO(3) – what's going on? It turns out that SU(2) is so similar to SO(3) that they have identical Lie algebras, i.e.

$$[X_i, X_j] = i\varepsilon_{ijk} X_k$$

We choose to use SU(2) because it is so important in higher dimensional Lie algebras and this is a perfect opportunity to work with it. SU(2) is the <u>S</u>pecial (determinant =1), <u>U</u>nitary $\left(A^\dagger A = AA^\dagger = I\right)$ group and <u>2</u> dimensional, complex matrices. Like SO(3), it has 3 parameters and also defines rotations. Below we have a representation for SU(2) which has all the required properties, i.e. $M\left(0,\,0,\,0\right) = I, \;\; \det\left(M\right) = 1, \;\; MM^\dagger = M^\dagger M = I$,

$$M\left(a_I,\, b_R,\, b_I\right) = \frac{1}{\sqrt{\det}} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \; where \; a = 1 + ia_I \;\; and \;\; b = b_R + ib_I$$
$$and \;\; \det = 1 + a_I^2 + b_R^2 + b_I^2 \tag{5.5.1}$$

To reveal the rotational nature of the group, we will first calculate generators from this representation (5.5.1) using equation (5.2.1) and then exponentiate those generators. Differentiating M with respect to $b_R,\; b_I,\; and\; a_I$ and evaluating the derivatives at $b_R = b_I = a_I = 0$, gives us

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad X_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \quad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

i.e. the three Pauli matrices. Taking the derivatives involves evaluating

$$\frac{d}{da_I}\left[\frac{1}{\sqrt{\det}}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}\right] = \underbrace{\frac{d}{da_I}\left[\frac{1}{\sqrt{\det}}\right]}_{=0}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} + \frac{1}{\sqrt{\det}}\frac{d}{da_I}\left[\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}\right]$$

and the first term is always zero.

Following convention, we take as our generators half of the σ_i's, i.e. $X_i = \frac{1}{2}\sigma_i$.

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma_1}{2}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sigma_2}{2}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma_3}{2} \tag{5.5.2}$$

where X' could be used but it doesn't seem necessary. This convention is used for this 2d representation because SU(2) is so often used for $1/2$ spin situations so it is convenient to have half integer eigenvalues. The Lie algebra properties are not affected by this factor although the structure constants need to be multiplied by the square root of the same factor.

That the SU(2) generators follow the same Lie algebra as SO(3) i.e.

$$[X_i, X_j] = i\varepsilon_{ijk}X_k \tag{5.5.3}$$

can be verified exactly as we did for SO(3) in section 5.4.

To perform the exponentiation we use the same scheme used in section 4.2 on X_1. We again expand the exponential in a full Taylor expansion.

$$e^{i\theta_1 X_1} = e^{i\theta_1\frac{\sigma_1}{2}} = I + (i\theta_1\frac{\sigma_1}{2}) + (i\theta_1\frac{\sigma_1}{2})^2/2! + (i\theta_1\frac{\sigma_1}{2})^3/3! + (i\theta_1\frac{\sigma_1}{2})^4/4! + \ldots$$

$$= I + (i\frac{\theta_1}{2}\sigma_1) + (i\frac{\theta_1}{2}\sigma_1)^2/2! + (i\frac{\theta_1}{2}\sigma_1)^3/3! + (i\frac{\theta_1}{2}\sigma_1)^4/4! + \ldots$$

This is exactly the same as the result in equation (4.2.1) except that θ is replaced with $\frac{\theta_1}{2}$ and X is replaced by σ_1. Noticing that, just like X in equation (4.2.1), even powers of σ_1 are I and odd powers of σ_1 are σ_1, we can immediately write the solution from equation (4.2.2) as

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\frac{\theta_1}{2}) + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin(\frac{\theta_1}{2}) = \begin{pmatrix} cos(\frac{\theta_1}{2}) & -i\,sin(\frac{\theta_1}{2}) \\ i\,sin(\frac{\theta_1}{2}) & cos(\frac{\theta_1}{2}) \end{pmatrix}$$

The results for X_2 *and* X_3 are obtained in a similar way giving

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\frac{\theta_2}{2}) + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin(\frac{\theta_2}{2}) = \begin{pmatrix} \cos(\frac{\theta_2}{2}) & \sin(\frac{\theta_2}{2}) \\ -\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\frac{\theta_3}{2}) + i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin(\frac{\theta_3}{2}) = \begin{pmatrix} \cos(\frac{\theta_3}{2}) + i\sin(\frac{\theta_3}{2}) & 0 \\ 0 & \cos(\frac{\theta_3}{2}) - i\sin(\frac{\theta_3}{2}) \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta_3}{2}} & 0 \\ 0 & e^{-i\frac{\theta_3}{2}} \end{pmatrix}$$

Before leaving SU(2), we give the 3 dimensional generators because we will need them later.

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad (5.5.4)$$

5.6. Summary of Main Points

The basic tools of calculating generators and exponentiation was extended from one parameter to many parameters which introduced the general Lie algebra describing the required relationships between the generators. These relationships must be satisfied if the representation obtained from exponentiating the generators is to reflect the group structure.

The specific Lie algebra for SO(3) was derived because that algebra will be used to analyze the angular momentum of multi-particle systems in the next chapter.

The SU(2) group was introduced and, having the same Lie algebra as SO(3), will be used for our angular momentum work since it plays an important role in the theory of Lie algebras also.

6. Angular Momentum of Multi-Particle Systems

We will discuss the origins of raising and lowering operators and fill in the details with the SU(2) group. These operators are critical when working with multi-particle systems.

How tensor products are used to develop representations for multi particle systems is discussed using a spin 1 particle and a spin 1/2 particle. It is shown that the representation is reducible and we derive two irreducible representations to reflect the two particle states. The details of those states are then developed using the highest weight method.

Lastly, the identical set of two particle states are used to model nucleons, pions and delta particles by associating specific particles with states rather than angular momentum values.

6.1. Raising and Lowering Operators

Raising and lowering operators are important in quantum mechanics and in the theory of Lie algebras. In fact, these operators arise in quantum mechanics because of Lie groups and other commutator-based relationships. In this section, we will discuss how these raising and lowering operators arise. They appear whenever operators have a commutation relationship of the following form,

$$[N, X] = cX \qquad (6.1.1)$$

If N has an eigenvector $|n\rangle$ which satisfies $N|n\rangle = n|n\rangle$, then we can apply NX to $|n\rangle$ and use (6.1.1) to obtain.

$$\begin{aligned} NX|n\rangle &= \{XN + [N,X]\}|n\rangle \\ &= XN|n\rangle + [N,X]|n\rangle \\ &= Xn|n\rangle + cX|n\rangle \\ &= (n+c)X|n\rangle \end{aligned}$$

Hence,

$$N|n\rangle = n|n\rangle \tag{6.1.2}$$

$$X|n\rangle \propto |n+c\rangle \tag{6.1.3}$$

$$N|n+c\rangle = (n+c)|n+c\rangle \tag{6.1.4}$$

So that the action of X *on* $|n\rangle$ is to create a new eigenvector *of* $N, |n+c\rangle$, with eigenvalue $n+c$. If c is positive, then X is a raising operator and if negative, it is a lowering operator.

6.2. Raising and Lowering Operators for SU(2)

We are going to create raising and lowering operators, $X^{\pm}$, for the SU(2) group from the generators above, $X_i = \frac{1}{2}\sigma_i$, as follows

$$X^{\pm} = (X_1 \pm iX_2)/\sqrt{2}\ ^{6} \tag{6.2.1}$$

which satisfy

$$[X_3, X^{\pm}] = \pm X^{\pm} \tag{6.2.2}$$

and

$$[X^+, X^-] = X_3 \tag{6.2.3}$$

We demonstrate that $[X_3, X^+] = X^+$ below

$$[X_3, X^+] = \frac{1}{\sqrt{2}}(\underbrace{[X_3, X_1]}_{=iX_2} + i\underbrace{[X_3, X_2]}_{=-iX_1}) = \frac{1}{\sqrt{2}}(iX_2 + i(-1)iX_1) = \frac{1}{\sqrt{2}}(X_1 + iX_2) = X^+$$

where the Lie algebra condition, (equation (5.5.3)), is used to simplify the commutators. The other equation of (6.2.2) and equation (6.2.3) can be demonstrated in a similar fashion.

Note that $[X_3, X^{\pm}] = \pm X^{\pm}$ is of the same form as equation (6.1.1) so that X^+ is a raising operator that adds 1 to any eigenvalue of X_3 while X^- is a lowering operator that subtracts 1 from any eigenvalue of X_3. You may recognize that in quantum mechanics these same relationships exist

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[6] Some texts do not include the $1/\sqrt{2}$. With or without, equation (6.2.2) will be satisfied. We include it so that equation (6.2.3) will also be satisfied.

between $J_1,\ J_2,\ J_3$ *and* $J^{\pm}$. These relationships are the same because angular momentum in quantum mechanics follows the SO(3) group which, as we saw above, shares the same Lie algebra with SU(2). Since we are going to be using these operators to explore the angular momentum of multi-particle states, we will switch to the $\boldsymbol{J}$ notation. It is desirable to remember, though, that these 'angular momentum' relationships and all that develops from them are actually Lie algebra relationships.

The notation that we will be using for angular momentum states is $\left| j, m \right\rangle$ where j is the total angular momentum and m is the component of angular momentum in the z (or 3) direction. We take

$$\left\langle j_i,\, m_l \right|\left| j_j,\, m_k \right\rangle = \delta_{i\,j}\delta_{l\,k}. \tag{6.2.4}$$

Note that j and m appear in very different ways in the Lie algebra formulation. The m *in* $\left| j, m \right\rangle$ corresponds to the n *in* $\left| n \right\rangle$ in equation (6.1.2), i.e. the eigenvalue of the operator J_3, m, corresponds to the eigenvalue of the operator N, n. Similarly, the ± 1 in equation (6.2.2) corresponds to the c in equation (6.1.1), i.e. the increment in the eigenvalue of J_3, ± 1, induced by its raising and lowering operators corresponds to the increment in the eigenvalue of N, c, induced by the its raising and lowering operators.

The j *in* $\left| j, m \right\rangle$, however, has no analogy in the development of the generalized raising and lowering operators of section 6.1 because that development occurs without anything that would limit the amount of raising and lowering that can happen. Whereas, j limits the amount of raising and lowering that can occur because m has to be between $-j$ *and* $+j$. The nature of that limiting lies in the dimensionality, d_{rep}, of the specific representation being used. This is because the raising and lowering operators create independent eigenvectors and, within that representation, the number of independent eigenvectors is limited to d_{rep}. Therefore, j is determined by the dimensionality of the representation being used. So, for example, in the 2d defining representation of SU(2) using the Pauli matrices, the value of j must be 1/2 because that permits two values of m, 1/2 and -1/2 given that the increment of change in the eigenvalue is ± 1. In a 3d representation of SU(2), the value of j must be 1 because that permits three values of m, 1, 0 and -1 and so on. Notice that the increment of change in m is the same for all representations whereas the actual values that m takes is different in different representations.

In summary, the key eigenvector/eigenvalue relationships are:

$$J^2\left| j, m \right\rangle = j(j+1)\left| j, m \right\rangle \text{[7]} \tag{6.2.5}$$

$$J_3\left| j, m \right\rangle = m\left| j, m \right\rangle \tag{6.2.6}$$

The raising and lowering operators are

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[7] We haven't discussed the J^2 operator yet but for completeness we include it here. We will see it in **Section 6.5**

$$J^{\pm} = \left(J_1 \pm iJ_2\right) / \sqrt{2} \tag{6.2.7}$$

Two key normalization relationships involving $J^{\pm}$ that we will need are:

$$J^{+}\left| j,\ m\right\rangle = \sqrt{(j+m+1)(j-m)/2}\left| j,\ m+1\right\rangle \tag{6.2.8}$$
$$J^{-}\left| j,\ m\right\rangle = \sqrt{(j+m)(j-m+1)/2}\left| j,\ m-1\right\rangle \tag{6.2.9}$$

The factors $\sqrt{(j+m+1)(j-m)/2}$ and $\sqrt{(j+m)(j-m+1)/2}$ are the constants of proportionality in equation (6.1.3). Equations (6.2.8) and (6.2.9) will be used to extract the irreducible representations from the tensor product which represents the multi-particle state. We derive equation (6.2.8) below and equation (6.2.9) is left as an exercise.

We begin with the equivalent of equation (6.1.3) with proportionality constant, k

$$J^{+}\left| j,\ m\right\rangle = k\left| j,\ m+1\right\rangle \tag{6.2.10}$$

and its adjoint

$$\left\langle j,\ m\right| J^{-} = \left\langle j,\ m+1\right| k^{*}$$

Now we multiply both LHSs and both RHSs to get

$$\left\langle j,\ m\right| J^{-}J^{+}\left| j,\ m\right\rangle = \left\langle j,\ m+1\right| k^{*}k\left| j,\ m+1\right\rangle$$
$$\left\langle j,\ m\right| J^{-}J^{+}\left| j,\ m\right\rangle = k^{*}k$$

Using equations (6.2.7) and some straightforward work and the Lie algebra condition (5.5.3) , $\left[J_1,\ J_2\right] = iJ_3$, we get

$$\left\langle j,\ m\right| \frac{1}{2}\left(J^2 - J_3^2 - J_3\right)\left| j,\ m\right\rangle = k^{*}k$$
$$\frac{1}{2}(j(j+1) - m^2 - m) = k^{*}k$$
$$\frac{1}{2}\left(j+m+1\right)\left(j-m\right) = k^{*}k$$
$$\left|k\right| = \sqrt{\left(j+m+1\right)\left(j-m\right)/2}$$

If you are following this derivation you may find that, rather than trying to derive $\left(j+m+1\right)\left(j-m\right)$ from $\left(j\left(j+1\right) - m^2 - m\right)$, it is easier to just show that they are equal. Inserting $\left|k\right|$ into equation (6.2.10), we get

$$J^{+}\left| j,\ m\right\rangle = \sqrt{\left(j+m+1\right)\left(j-m\right)/2}\left| j,\ m+1\right\rangle$$

thus, verifying equation (6.2.8)

6.3. An Alternative Method of Calculating Generators

This section, although interesting and useful, does not really fit in the sequence of the other sections in this chapter. It is put here because the introduction of states, $|\, j,\ m\rangle$, in section 6.2 is a prerequisite to understanding this alternative method. The section is included because it helps to illustrate the relationship between the states of the algebra and the operators of the algebra (i.e. the generators). What is done here is to derive the generators from chosen basis states and chosen basis vectors. The natural choice of the basis states for SU(2) for the 2d representation is $|1/2,\ 1/2\rangle\ \ and\ \ |1/2,\ -1/2\rangle$.with the basis vectors being

$$\begin{pmatrix}1\\0\end{pmatrix}\ \ and\ \ \begin{pmatrix}0\\1\end{pmatrix}\text{ respectively.}$$

We know that

$$J_3\begin{pmatrix}1\\0\end{pmatrix}=1/2\begin{pmatrix}1\\0\end{pmatrix}\ \ and\ \ J_3\begin{pmatrix}0\\1\end{pmatrix}=-1/2\begin{pmatrix}0\\1\end{pmatrix}$$

because of equation (6.2.6) so

$$J_3=\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}$$

identical to equation (5.5.2). To find $J_1\ \ and\ \ J_2$, we must first find $J^+\ \ and\ \ J^-$. We know that

$$J^+\begin{pmatrix}1\\0\end{pmatrix}=0\ \ and\ \ J^+\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$$

from equation (6.2.8). As a result,

$$J^+=\begin{pmatrix}0 & \frac{1}{\sqrt{2}}\\ 0 & 0\end{pmatrix}.$$

Similarly,

$$J^-\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\ \ and\ \ J^-\begin{pmatrix}0\\1\end{pmatrix}=0$$

so

$$J^-=\begin{pmatrix}0 & 0\\ \frac{1}{\sqrt{2}} & 0\end{pmatrix}$$

Using $J_1=\sqrt{\frac{1}{2}}\left(J^++J^-\right)\ \ and\ \ J_2=-i\sqrt{\frac{1}{2}}\left(J^+-J^-\right)$ we calculate $J_1\ \ and\ \ J_2$ as

$$J_1=\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}\ \ and\ \ J_2=\begin{pmatrix}0 & -i/2\\ i/2 & 0\end{pmatrix}$$

identical to equation (5.5.2). A different choice of the basis states or the basis vectors would give a different representation of the generators.

6.4. Tensor Products and Direct Sums

We'll now develop the machinery for combining two (or more) representations in order to model two particles with angular momentum. When the two representations are combined we usually obtain a reducible representation. To gain useful information from it, we need to decompose it into its irreducible components.

For our purposes, a tensor is just a square matrix of a certain dimension. A tensor product allows us to multiply two matrices (which can have different dimensions) together. The purpose of the product is to delineate all possibilities of how the two can be combined. An example of a tensor product of a 3x3 matrix and a 2x2 matrix follows.

$$\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\otimes\begin{pmatrix}1&2\\0&3\end{pmatrix}=\begin{pmatrix}1\begin{pmatrix}1&2\\0&3\end{pmatrix}&0\begin{pmatrix}1&2\\0&3\end{pmatrix}&0\begin{pmatrix}1&2\\0&3\end{pmatrix}\\0\begin{pmatrix}1&2\\0&3\end{pmatrix}&2\begin{pmatrix}1&2\\0&3\end{pmatrix}&0\begin{pmatrix}1&2\\0&3\end{pmatrix}\\0\begin{pmatrix}1&2\\0&3\end{pmatrix}&0\begin{pmatrix}1&2\\0&3\end{pmatrix}&3\begin{pmatrix}1&2\\0&3\end{pmatrix}\end{pmatrix}=\left(\begin{array}{cc:cc:cc}1&2&0&0&0&0\\0&3&0&0&0&0\\\hdashline 0&0&2&4&0&0\\0&0&0&6&0&0\\\hdashline 0&0&0&0&3&6\\0&0&0&0&0&9\end{array}\right)$$

It is clear from the example that the dimension of the tensor product, $d_{1\otimes 2}$, is the product of the dimensions of the two original matrices, i.e. $d_{1\otimes 2}=d_1\ x\ d_2$.

States from space 1 and space 2 also need to be combined so that $|1\rangle$ *and* $|2\rangle$ combine into $|1\rangle|2\rangle$ *or* $|1\rangle\otimes|2\rangle$ *or* $|1\ 2\rangle$. Most of the time we will use $|1\rangle|2\rangle$ to show that the kets are separate (even though adjacent) and are operated on by operators that are either in space 1 or space 2.

When we are performing tensor products with Lie groups/algebras we need to be careful because the matrices representing the group elements follow the pattern of the example above but the matrices representing the generators of the Lie algebra do not. The reason for this is that the matrices representing the group elements interact with one another through multiplication whereas the matrices representing the generators interact with one another through addition since they reflect the group elements only after exponentiation.

If we are taking the tensor product between a 3 dimensional space and a 2 dimensional space then the result will be a 6 dimensional space. The associated generators that will be added are also 3 dimensional and 2 dimensional and the final result will also be 6 dimensional. To add these two generators correctly, we need to recognize that the first generator only operates on the first ket of the tensor product while the second generator only operates on the second ket. We can obtain this result by forming the tensor product of the first generator (3d) with a 2d identity matrix forming a 6d matrix. Thus the first generator operates on the first ket while the second ket remains unchanged because of the identity matrix. Then, we form the tensor product of a 3d identity matrix with the second generator (2d) also forming a 6d matrix. These two 6d matrices are then added to form the generator of the tensor product of the two original spaces. An example of this construction is shown in equation (6.4.1)

The first step in creating the J_3 for the 2 particle states is to combine J_3^1, a 3 x 3 matrix (equation (5.5.4)), and $J_3^{1/2}$, a 2 x 2 matrix (equation (5.5.2)), and form the generator of their tensor product.

$$J_3^1 \otimes I_{2x2} + I_{3x3} \otimes J_3^{1/2} \tag{6.4.1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 1\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & 0\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & 0\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \\ 0\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & 1\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & 0\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \\ 0\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & 0\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & 1\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 & & & & & \\ & 1/2 & & & & \\ & & 1/2 & & & \\ & & & -1/2 & & \\ & & & & -1/2 & \\ & & & & & -3/2 \end{pmatrix} \rightarrow \left(\begin{array}{cccc|cc} 3/2 & & & & & \\ & 1/2 & & & & \\ & & -1/2 & & & \\ & & & -3/2 & & \\ \hline & & & & 1/2 & \\ & & & & & -1/2 \end{array}\right) \tag{6.4.2}$$

By rearranging the eigenvalues along the diagonal and partitioning the result we see that we need 2 irreducible representations, $J_3^{3/2}$ *and* $J_3^{1/2}$, to describe the angular momentum states of the 2 particle system. The direct sum expresses the relationship of the two irreducible representations to the original representations, J_3^1 *and* $J_3^{1/2}$

$$J_3^{1\otimes 1/2} = J_3^{3/2} \oplus J_3^{1/2}.$$

6.5. Extract the Irreducible Representations

We will use the **highest weight method** to extract the irreducible representations from the tensor product. This involves identifying the **highest weight state (HWS)** of a representation and using the lowering operator to recursively find all the other states. The HWS will have

1. The largest j value possible and
2. The value of m will equal that largest j.

The state that has this property, for our example, is $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ which is $\left|1, 1\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle$.

The term highest weight state will make more sense later when we discuss Lie algebras in more detail but for now we can just say that weights of a Lie algebra's representation are eigenvalues of some of the generators. In the case of SU(2), these eigenvalues are the various m values that occur. When the tensor product of representations is being considered, it is the combined state with the largest possible j value and $m = j$ that we need to identify.

We start with the HWS

$$\boldsymbol{\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \left|1, 1\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle} \tag{6.5.1}$$

.

for the highest weight state. (We use **bolding** to highlight the states that we will be identifying.) So now we apply J^- to this state and obtain

$$J^-\left|\frac{3}{2}, \frac{3}{2}\right\rangle = J^-\left[\left|1, 1\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right]$$

$$J^-\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \underbrace{\left[J^-\left|1, 1\right\rangle\right]}_{\downarrow}\underbrace{\left|\frac{1}{2}, \frac{1}{2}\right\rangle}_{\downarrow} + \underbrace{\left|1, 1\right\rangle}_{\downarrow}\underbrace{J^-\left|\frac{1}{2}, \frac{1}{2}\right\rangle}_{\downarrow} \tag{6.5.2}$$

$$\overbrace{J_A^-\left|\ A\ \right\rangle}^{\uparrow}\otimes\overbrace{I_B\left|\ B\ \right\rangle}^{\uparrow} + \overbrace{I_A\left|\ A\ \right\rangle}^{\uparrow}\otimes\overbrace{J_B^-\left|\ B\ \right\rangle}^{\uparrow} \tag{6.5.3}$$

Expression (6.5.3) is a graphic display of equation (6.4.1) lined up with equation (6.5.2) above to show how the upper equation was arrived at. In expression (6.5.3) the $'A'$ designates the first particle space (3d) and the $'B'$ designates the second particle space (2d).

Note that, in equation (6.5.2), although the $J^-'s$ are all lowering operators, they are not all the same because they operate on spaces of different dimensionality. The J^- on the LHS operates on states in the tensor product space and thus is operating on 6d kets. The first J^- on the RHS operates on states of the first particle and thus operates on 3d kets. The second J^- on the RHS operates on the states of the second particle and thus operates on 2d kets. The reason that we can show all of them as being just J^- is because the Lie algebra has relationships independent of representation, i.e. the relationships are true in all representations. Equation (6.2.9)

$$J^-\left|\ j,\ m\right\rangle = \sqrt{(j+m)(j-m+1)/2}\left|\ j,\ m-1\right\rangle \tag{6.5.4}$$

is one of those relationships that is true regardless of the dimensionality of the representation.

Thus, it does not add anything to label each J^- with the dimensionality that it operates in to take advantage of equation (6.5.4). This is the second relationship that we have encountered that is independent of representation – the first was equation (5.3.3).

We use (6.5.4) to simplify (6.5.2). The LHS of (6.5.2) gives

$$J^-\left|\frac{3}{2},\frac{3}{2}\right\rangle = \sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{3}{2}+1)/2}\left|\frac{3}{2},\ \frac{3}{2}-1\right\rangle = \sqrt{\frac{3}{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle$$

The RHS of (6.5.2) is

$$[J^-|1,\ 1\rangle]\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + |1,1\rangle J^-\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$

Using (6.5.4)

$$=\sqrt{(2)(1)/2}\,|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + |1,\ 1\rangle\sqrt{(1)(1)/2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$=|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + \sqrt{1/2}\,|1,\ 1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Therefore, setting LHS = RHS,

$$\sqrt{3/2}\left|\frac{3}{2},\ \frac{1}{2}\right\rangle = |1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + \sqrt{1/2}\,|1,\ 1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

and finally we get

$$\boldsymbol{\left|\frac{3}{2},\ \frac{1}{2}\right\rangle = \sqrt{2/3}\,|1,0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + \sqrt{1/3}\,|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle} \qquad (6.5.5)$$

As hoped, the normalization of $\left|\frac{3}{2},\ \frac{1}{2}\right\rangle$ is 1 since $\left(\sqrt{2/3}^2+\sqrt{1/3}^2\right)^{1/2}=1$.

Applying J^- to the $\left|\frac{3}{2},\ \frac{1}{2}\right\rangle$ state, equation (6.5.5) , yields the next lower state

$$J^-\left|\frac{3}{2},\ \frac{1}{2}\right\rangle = \sqrt{2/3}\left[J^-|1,\ 0\rangle\right]\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + \sqrt{2/3}\,|1,\ 0\rangle J^-\left|\frac{1}{2},\frac{1}{2}\right\rangle$$
$$+\sqrt{1/3}\left[J^-|1,\ 1\rangle\right]\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{1/3}\,|1,\ 1\rangle\underbrace{J^-\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle}_{=0}$$

$$\sqrt{2}\left|\frac{3}{2},\ -\frac{1}{2}\right\rangle = \sqrt{2/3}\,|1,\ -1\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + \sqrt{2/3}\,|1,\ 0\rangle\sqrt{1/2}\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$+\sqrt{1/3}\,|1,\ 0\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle\ +\ 0$$

The fourth term of the RHS is 0 because $J^-\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = 0$ since $\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$ is the lowest weight state of $j=\frac{1}{2}$ and cannot be lowered further. Finally, we get

$$\mathbf{\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{1/3}\,|1,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\ +\ \sqrt{2/3}\,|1,0\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle}. \qquad (6.5.6)$$

We apply J^- to the $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$ state to complete the process of finding all the states in the $j=\frac{3}{2}$ representation.

$$J^-\left|\frac{3}{2},\ -\frac{1}{2}\right\rangle = \sqrt{1/3}\overbrace{\left[J^-|1,-1\rangle\right]}^{=0}\left|\frac{1}{2},\ \frac{1}{2}\right\rangle\ +\ \sqrt{1/3}\,|1,\ -1\rangle J^-\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$+\sqrt{2/3}[J^-|1,\ 0\rangle]\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle\ +\ \sqrt{2/3}\,|1,\ 0\rangle\underbrace{J^-\left|\frac{1}{2},-\frac{1}{2}\right\rangle}_{=0}$$

$$\sqrt{3/2}\left|\frac{3}{2},-\frac{3}{2}\right\rangle = 0\ +\ \sqrt{1/3}\,|1,\ -1\rangle\sqrt{1/2}\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$
$$+\sqrt{2/3}\,|1,\ -1\rangle|1/2,\ -1/2\rangle\ +\ 0$$

This time there are two 0's because $|1,\ -1\rangle$ is the lowest state of the $j=\frac{3}{2}$ representation and $\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$ is the lowest state of the $j=\frac{1}{2}$ representation. Finally, as we might expect,

$$\mathbf{\left|\frac{3}{2},-\frac{3}{2}\right\rangle = |1,-1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle} \qquad (6.5.7)$$

These four states, equations (6.5.1), (6.5.5), (6.5.6) and (6.5.7), are the four states corresponding to the $j=\frac{3}{2}$ representation. Note that these same four states could have been identified by starting with the lowest weight state, $\left|\frac{3}{2},-\frac{3}{2}\right\rangle = |1,\ -1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$, and applying J^+ recursively.

As we saw in matrix (6.4.2), there are a total of 6 states when we combine the spin 1 particle with the spin $1/2$ particle. We still have two states to identify which we expect will belong to the $j=\frac{1}{2}$ representation. We know that the states are orthogonal to each other so we need to find two states that are orthogonal to the four we have already defined. Orthogonality to $\left|\frac{3}{2},\ \frac{3}{2}\right\rangle$ and $\left|\frac{3}{2},\ -\frac{3}{2}\right\rangle$ is assured because no other states can include the highest or lowest states in their composition.

We will assume a state orthogonal to equation (6.5.5) will have the form

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = A|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle + B|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle \tag{6.5.8}$$

We will find A and B by requiring orthogonality and normality: Orthogonality by multiplying equation (6.5.5) and the adjoint of equation (6.5.8).

$$\overbrace{\left\langle\frac{1}{2},\frac{1}{2}\middle|\middle|\frac{3}{2},\ \frac{1}{2}\right\rangle}^{=0} = \left\langle\frac{1}{2},\frac{1}{2}\right|\langle 1,\ 0|\,A^{\dagger}\sqrt{2/3}\,|1,\ 0\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left\langle\frac{1}{2},-\frac{1}{2}\right|\langle 1,1|\,B^{\dagger}\sqrt{1/3}\,|1,1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

giving

$$0 = A^{\dagger}\sqrt{2/3} + B^{\dagger}\sqrt{1/3}.$$

And normality by multiplying equation (6.5.8) with its adjoint.

$$\overbrace{\left\langle\frac{1}{2},\ \frac{1}{2}\middle|\middle|\frac{1}{2},\ \frac{1}{2}\right\rangle}^{=1} = A^{\dagger}A + B^{\dagger}B = 1.$$

$A = \sqrt{1/3}$ *and* $B = -\sqrt{2/3}$ satisfy these conditions (although the choice of + - rather than - + is arbitrary) so we take

$$\boldsymbol{\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{1/3}\,|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle\ -\ \sqrt{2/3}\,|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle} \tag{6.5.9}$$

We continue the use of **bolding** for highlighting the states of interest. Similarly, a state orthogonal to equation (6.5.6) is

$$\boldsymbol{\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = \sqrt{2/3}\,|1,\ -1\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle\ -\ \sqrt{1/3}\,|1,\ 0\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle} \tag{6.5.10}$$

We now show that the choice between + or – in equation (6.5.10) is not arbitrary but rather determined by the choice made for the previous state, equation (6.5.9). We show this by applying the highest weight method to the 2d subspace associated with the $j=\frac{1}{2}$ representation. The highest weight state is (6.5.9) since it is the state with $m = j$.

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{1/3}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle\ -\ \sqrt{2/3}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

Applying J^- as we did above to the 4d subspace associated with the $j=\frac{3}{2}$ representation, we obtain

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{2/3}\left|1,\ -1\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle - \sqrt{1/3}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

which is identical to the second state (6.5.10) that we identified using orthogonality and normality arguments. This tells us that although the first choice of + - or - + was arbitrary, the second choice is not since the final state needs to follow from the highest weight method as well as be orthonormal. In fact, as we saw earlier the highest weight method assures orthonormality.

In the next section we will develop the Casimir operator, J^2, and verify that the j value of the first of the two new states, equation (6.5.9) is $\frac{1}{2}$. Showing that the second of the two new states, equation (6.5.10), also has j=1/2 follows the same procedure.

6.6. J^2 Verifies the j Values

When we assigned j values to the states during the use of the highest weight method, we are using deductions to make those assignments. It would be gratifying if we could 'measure' the j value of a state and verify that our deductions are correct. The Casimir operator, J^2, provides such a measure. A Casimir operator is one which commutes with all the generators. For the SU(2) group there is only one Casimir operator

$$J^2\left|\ j,\ m\right\rangle = (J_1^2 + J_2^2 + J_3^2)\left|\ j,\ m\right\rangle = j(j+1)\left|\ j,\ m\right\rangle.$$

We need to derive a form of J^2 that we can apply to our state (6.5.9)

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{1/3}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle\ -\ \sqrt{2/3}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle \qquad (6.6.1)$$

The issue is that we do not know how J_1 *and* J_2 act on our states so they need to be replaced by J^+ *and* J^-. In the derivation below, we revise our usual notation slightly to make the equations easier to read – we have moved the *j* values down from the superscripts of J to the subscripts so that the *j* values do not confuse squaring or the $\pm$ of the raising and lowering operators.

$$J_{3/2}^2 = (J_1 + J_{1/2})^2 = J_1^2 + J_{1/2}^2 + 2J_1J_{1/2}$$

$$= J_1^2 + J_{1/2}^2 + 2(J_{1x}J_{1/2x} + J_{1y}J_{1/2y} + J_{1z}J_{1/2z})$$
$$= J_1^2 + J_{1/2}^2 + 2J_{1z}J_{1/2z} + 2(J_{1x}J_{1/2x} + J_{1y}J_{1/2y})$$
$$= J_1^2 + J_{1/2}^2 + 2J_{1z}J_{1/2z} + 2(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+) \qquad (6.6.2)$$

Showing that

$$J_{1x}J_{1/2x} + J_{1y}J_{1/2y} = J_1^+J_{1/2}^- + J_1^-J_{1/2}^+ \qquad (6.6.3)$$

in the last step, is straightforward so it is left as an exercise. In some texts, a factor of 2 appears in the RHS of (6.6.3) because slightly different definitions of J^+ *and* J^- are used. I only mention this because it confused me until I understood why. The footnote to equation (6.2.1) addresses this issue.

We now apply the J^2 operator derived above, equation (6.6.2), to state (6.6.1)

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{1/3}\,|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle - \sqrt{2/3}\,|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$J_{3/2}^2\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \left[J_1^2 + J_{1/2}^2 + 2J_{1z}J_{1/2z} + 2(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+)\sqrt{1/3}\right]|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$-\left[J_1^2 + J_{1/2}^2 + 2J_{1z}J_{1/2z} + 2\left(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+\right)\right]\sqrt{2/3}\,|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \left[(1)(1+1) + (1/2)(1/2+1) + 2(0)(1/2) + 2\left(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+\right)\right]\sqrt{1/3}\,|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$-\left[(1)(1+1) + (1/2)(1/2+1) + 2(1)(-1/2) + 2\left(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+\right)\right]\sqrt{2/3}\,|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \left[2 + 3/4 + 0 + 2\left(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+\right)\right]\sqrt{1/3}\,|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$-\left[2 + 3/4 + (-1) + 2\left(J_1^+J_{1/2}^- + J_1^-J_{1/2}^+\right)\right]\sqrt{2/3}\,|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \sqrt{1/3}\left(11/4\right)|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle \ + \ 2\sqrt{1/3}\left(J_1^+J_{1/2}^- + \underbrace{J_1^-J_{1/2}^+}_{on\ |1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = 0}\right)|1,\ 0\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$

$$-\sqrt{2/3}\left(7/4\right)|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle \ - \ 2\sqrt{2/3}\left(\underbrace{J_1^+J_{1/2}^-}_{on\ |1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = 0} + J_1^-J_{1/2}^+\right)|1,\ 1\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

In applying the raising and lowering operators we note that two of the four terms are zero because $J_{1/2}^{+}\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$ is 0 and $J_1^{+}\left|1,\ 1\right\rangle$ is 0. The non-zero terms give

$$= \sqrt{1/3}\left(11/4\right)\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle \ + \ 2\sqrt{1/3}J_1^{+}J_{1/2}^{-}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$-\sqrt{2/3}(7/4)\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle \ - \ 2\sqrt{2/3}J_1^{-}J_{1/2}^{+}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \sqrt{1/3}\left(11/4\right)\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle \ + \ 2\sqrt{1/3}\left(1\right)\left|1,\ 1\right\rangle\sqrt{1/2}\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$
$$-\sqrt{2/3}\left(7/4\right)\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle \ - \ 2\sqrt{2/3}\left(1\right)\left|1,\ 0\right\rangle\sqrt{1/2}\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$

Grouping the terms by state gives

$$= \sqrt{1/3}\left(11/4\right)\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle \ - \ 2\sqrt{1/3}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$-\sqrt{2/3}\left(7/4\right)\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle \ + \ \sqrt{2/3}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \left\{\sqrt{1/3}\left(11/4\right) - 2\sqrt{1/3}\right\}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$\left\{-\sqrt{2/3}\left(7/4\right) + \sqrt{2/3}\right\}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \sqrt{1/3}\left\{\left(11/4\right) - 2\right\}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$
$$-\sqrt{2/3}\left\{\left(7/4\right) - 1\right\}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle$$

$$= \left(3/4\right)\underbrace{\left\{\sqrt{1/3}\left|1,\ 0\right\rangle\left|\frac{1}{2},\ \frac{1}{2}\right\rangle \ - \ \sqrt{2/3}\left|1,\ 1\right\rangle\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle\right\}}_{=\left|\frac{1}{2},\ \frac{1}{2}\right\rangle}$$

$$= (3/4)\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \underbrace{(1/2)(1/2+1)}_{=j(j+1)}\left|\frac{1}{2},\ \frac{1}{2}\right\rangle$$

As expected, the j value for $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ is $\frac{1}{2}$.

6.7. Isospin and the Decay of the Δ^+ Particle

Angular momentum and spin are both rotational phenomenon in space-time. The study of elementary particles has led to the understanding that there are other kinds of rotational phenomenon that occur in internal spaces which share the same symmetries as those of ordinary rotation. Isospin is such an internal space. It arises because of charge independence of the strong force i.e. that nucleons and pions appear to interact without regard to the electric charges of the interacting particles (at least approximately).

This is expressed as the nucleons, i.e. the proton and the neutron, form a doublet and they are viewed as being essentially the same particle having two isospin states, $I_3 = +\frac{1}{2} \text{ and } -\frac{1}{2}$ which can be transformed into one another through a kind of rotation. The nucleons manifest the 2 dimensional representation of SU(2) with $I = \frac{1}{2}$. Similarly, the pions, i.e. $\pi^+, \pi^0 \text{ and } \pi^-$, form a triplet and are viewed as being essentially the same particle having three isospin states, $I_3 = +1, 0 \text{ and } -1$. They manifest the 3 dimensional representation of SU(2) with $I = 1$

If we scatter pions off nucleons we are creating a multi-particle state where now we are interested in the combinations of isospin. It is amazing that exactly the same mathematical structure that we created to study the angular momentum of a multi-particle system can be used to study the scattering of pions off nucleons! We know from our analysis of spin that we have two irreducible representations of the 2 particle states – one is the 4 dimensional representation of SU(2) with $I = \frac{3}{2}$. And the other is the 2 dimensional representation of SU(2) with $I = \frac{1}{2}$.

Experimentally, they have found four resonances in the scattering of pions off nucleons which correspond to a multiplet of 4 particles called $\Delta^{+2}, \Delta^{+1}, \Delta^0 \text{ and } \Delta^{-1}$ which manifest the 4 dimensional representation of SU(2) with $I = \frac{3}{2}$. Isospin, like angular momentum and spin, is conserved so we can delineate how the 4 resonances decay into the pions and the nucleons in **Table 6.7.1** below.

Table 6.7.1 – Decay Modes of Δ particles

Delta	I_3	Delta State	Pion	Pion State	Nucleon	Nucleon State	Combined State
Δ^{++}	$\frac{3}{2}$	$\left\|\frac{3}{2}, \frac{3}{2}\right\rangle$	π^+	$\|1, 1\rangle$	p	$\left\|\frac{1}{2}, \frac{1}{2}\right\rangle$	$\|1, 1\rangle\left\|\frac{1}{2}, \frac{1}{2}\right\rangle$
Δ^{+}	$\frac{1}{2}$	$\left\|\frac{3}{2}, \frac{1}{2}\right\rangle$	π^+ π^0	$\|1, 1\rangle$ $\|1, 0\rangle$	n p	$\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$ $\left\|\frac{1}{2}, \frac{1}{2}\right\rangle$	$\|1, 1\rangle\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$ $\|1, 0\rangle\left\|\frac{1}{2}, \frac{1}{2}\right\rangle$
Δ^{0}	$-\frac{1}{2}$	$\left\|\frac{3}{2}, -\frac{1}{2}\right\rangle$	π^0 π^-	$\|1, 0\rangle$ $\|1, -1\rangle$	n p	$\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$ $\left\|\frac{1}{2}, \frac{1}{2}\right\rangle$	$\|1, 0\rangle\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$ $\|1, -1\rangle\left\|\frac{1}{2}, \frac{1}{2}\right\rangle$
Δ^{-}	$-\frac{3}{2}$	$\left\|\frac{3}{2}, -\frac{3}{2}\right\rangle$	π^-	$\|1, -1\rangle$	n	$\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$	$\|1, -1\rangle$ $\left\|\frac{1}{2}, -\frac{1}{2}\right\rangle$

At this point the group theoretic approach has described the nucleon doublet, the pion triplet and the Δ quartet. In addition, states have been assigned to all the decay paths. With this information we can compare, for example, the two decay paths for the Δ^+ resonance to determine their relative probabilities. This is possible because of our work in the previous section leading to equation (6.5.5).

.

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{2/3}\,|1, 0\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{1/3}\,|1, 1\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \qquad (6.7.1)$$

We calculate the amplitude of decay to π^+ *and* n by multiplying equation (6.7.1), the total state for Δ^+, with the adjoint of $|1, 1\rangle|1/2, -1/2\rangle$, the combined state for the $\pi^+ n$ decay mode.

$$
\begin{aligned}
\left\langle \pi^{+} n \middle\| \Delta^{+} \right\rangle &= \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \langle 1, 1 | \middle| \frac{3}{2}, \frac{1}{2} \right\rangle \\
&= \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \langle 1, 1 | \left\{ \underbrace{\sqrt{2/3}\,|1, 0\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \sqrt{1/3}\,|1, 1\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle}_{\textit{from equation } (6.7.1)} \right\} \\
&= \sqrt{2/3} \underbrace{\left\langle \frac{1}{2}, -\frac{1}{2} \middle| \langle 1, 1 \| 1, 0 \rangle \middle| \frac{1}{2}, \frac{1}{2} \right\rangle}_{=0} + \sqrt{1/3} \underbrace{\left\langle \frac{1}{2}, -\frac{1}{2} \middle| \langle 1, 1 \| 1, 1 \rangle \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle}_{=1} \\
&= \sqrt{1/3}
\end{aligned}
\tag{6.7.2}
$$

Similarly, for the decay to π^0 *and* p

$$
\left\langle \pi^{0} p \middle\| \Delta^{+} \right\rangle = \left\langle \frac{1}{2}, \frac{1}{2} \middle| \langle 1, 0 | \middle| \frac{3}{2}, \frac{1}{2} \right\rangle = \sqrt{2/3}
$$

Therefore the relative amplitudes are

$$
\frac{\left\langle \pi^{0} p \middle\| \Delta^{+} \right\rangle}{\left\langle \pi^{+} n \middle\| \Delta^{+} \right\rangle} = \sqrt{2}
$$

Since decay probability is proportional to the amplitude squared, we learn that the decay probability for the $\pi^0 p$ path is twice that of the $\pi^+ n$ path. And this result, verified by experiment, is obtained from group theoretic principles.

6.8. Summary of Main Points

Raising and lowering operators were introduced and shown to play a key role in highest weight method for breaking a reducible representation down into irreducible representations. They were shown to arise when a commutation relationship of the form $[N, X] = cX$ exists.

Multi-particle states are modeled with tensor products of representations resulting in a reducible representation. An example of spin 1 and spin 1/2 particles is used to demonstrate how this reducible representation can be broken down into two irreducible representations.

The highest weight method was used to identify the various multi-particle states that exist within each irreducible representation.

The J^2 operator was used to confirm the j values for certain states.

The same exact framework for multi-particle states was used to describe the n, p doublet, the π^+, π^0, π^- triplet and the $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$ quartet. Relative decay rates were calculated for Δ^+ using group theoretic principles only.

7. Introduction to Roots and Weights

We have seen that the generators of the Lie algebra provide a compact description of the Lie algebra and consequently also of the Lie group. In addition, different representations of these generators have permitted us to analyze multi-particle states – first, we considered the angular momentum of multi-particle states and then, the existence of related groups of particles, multiplets, and we even extracted some relative decay rate information.

Now we will go deeper into the Lie algebra and extract an even more compact expression of the algebra – Simple roots. The simple roots have the same property as the generators in that the entire Lie algebra/group can be reconstructed from them. The advantage of using the simple roots is that there are far fewer of them. For the SU(10) group, for example, there are 99 generators and only 9 simple roots. Using simple roots to construct irreducible representations, for example, is easier than using generators.

We begin by describing the **adjoint representation** (of the generators) which define the roots of the algebra. Corresponding elements called weights exist in other representations and we will usually use the **defined representation** as sort of a surrogate for other, non-adjoint representations.

This chapter may, in fact, not be relevant for many readers. It depends on whether their course of study includes this deeper look into Lie algebras or not.

7.1. The Adjoint Representation

There is a particular representation of the generators, the adjoint representation, which plays a special role in the theory of roots. There does not seem to be any consistent nomenclature for the adjoint representation so we will use A.

Starting with the general expression of a Lie algebra from equation (5.3.3)

$$[X_i, X_j] = iC_{ij}^k X_k,$$

we define the adjoint representation as the set of matrices that are derived directly from the structure constants

$$A_{jk}^i = -iC_{ij}^k$$

If the dimension of the Lie algebra (i.e. the number of generators) is d, then i, j and k all vary from 1 to d. Therefore, the dimension of the matrices A^i is d and the number of matrices is also d. Since the C_{ij}^k's are real, the $A's$ are pure imaginary, anti-symmetric and therefore Hermitian with real eigenvalues.

What makes the adjoint representation special? All representations have one matrix for each generator but only the adjoint representation has the same index for the rows and the columns of those matrices as the index for the generators themselves. That means that the states of the adjoint representation, column vectors, correspond to the generators themselves and to the matrices which represent the generators. We will see shortly that this correspondence is important for the theory of Lie algebras.

7.2. The Defining Representation

We will use the defining representation as a surrogate for all non-adjoint representations. The results that we obtain for the defining representation will be representative of the results that we would obtain using another non-adjoint representation. In order to define the defining representation, we will first need to define a **fundamental representation.**

A fundamental representation is a representation whose highest weight is a fundamental weight. The number of fundamental representations is equal to the rank of the Li algebra which will be discussed shortly. Unfortunately, we haven't talked formally about weights yet so this definition doesn't help us much. Coming at it from a different angle, we note that each Lie algebra and its generators have a smallest dimensionality that can support a matrix representations of those generators. For the SU(N) and SO(N) groups, the ones in which we are the most interested, that smallest dimensionality is N. We can select as our basis for this N dimensional space, the simple basis vectors $(1,0,0,...)$, $(0,1,0,...)$,$(0,0,1,...)etc$. The matrices in this basis which represent the generators are collectively called the fundamental representation in N space. There are $N-2$ more fundamental representations in higher dimensional spaces that we will not deal with.

The defining representation is the fundamental representation in the smallest dimensionality. It is sometimes referred to as the fundamental representation but since there are more than one fundamental representations it is clear that this usage is referring to the smallest dimension fundamental representation.

7.3. The Cartan Basis of a Lie Algebra

In the set of all the $A's$, there are specific $A's$ which commute with one another and which, therefore, can be diagonalized simultaneously. Since, in physics, we want to diagonalize as much as possible, we want the largest set of commuting $A's$. These define the **Cartan sub-algebra**. These matrices we will designate as $H's$ so

$$H_i = A_i \quad with \quad i=1 \quad to \quad r$$

where r, referred to as the **rank** of the algebra, is the number of commuting matrices. (We assume the $A's$ are numbered so that the commuting matrices are 1 to r.) The $H_i\text{'}s$ satisfy

$$H_i = H_i^\dagger \quad and \quad [H_i, H_j] = 0. \tag{7.3.1}$$

Linear combinations of the remaining $d-r$ $A's$ are referred to as $E's$ and they are selected so that

$$[H_i, E_{\alpha_j}] = (\alpha_{\underline{j}})_i E_{\alpha_{\underline{j}}} \ with \ j=1 \ to \ d-r \tag{7.3.2}$$

Where $\underline{j}$ indicates that the Einstein summing convention is not to be used. This selection of the $E's$ is possible because the $H's$ are all taken as diagonal. Notice that each of the α_j subscripts for the $E's$ are actually vectors. Thus, E_{α_j} is a single operator but when commuted with each of the $H_i's$ generates a different eigenvalue, $(\alpha_j)_i$. Most texts do not use the j in their notation. We do, to make it explicit that there are multiple $\alpha's$. The problem (for me) with leaving out the j is that the general $E, E_\alpha,$ in the regular notation has exactly the same 'name' as the first element in the series of $E's - E_\alpha, E_\beta, E_\gamma,...$ For the $H's$ this does not occur since the general H is H_i and the series of $H's$ *is* $H_1, H_2, H_3,...$ as is usually done. Thus, by adding the j we get the same specificity with the $E's$ that we already have with the $H's$. The series of $E's$ then becomes $E_{\alpha_1}, E_{\alpha_2}, etc.$ Strictly speaking, we could drop the α but that seems like an unnecessary break from tradition! The need to add the underlining of j is considered to be an acceptable disadvantage.

Note that equation (7.3.2) has the same structure as equation (6.1.1) which tells us that the $E_{\alpha_j}'s$ are raising and lowering operators. This will be discussed in the next section.

The $(\alpha_j)_i's$ in equation (7.3.2) are known as the **root**s. The roots define a 2 dimensional array where there are r columns corresponding to the $H_i's$ and $d-r$ rows corresponding to the $E_{\alpha_j}'s$. Each row defines what is called a root vector of $1\ x\ r$ dimension, and there are $d-r$ root vectors. Thus, each α_j is a root vector.[8]

Root values always occur in pairs. We can see how this pairing arises if we take the adjoint of equation (7.3.2).

$$[H_i, E_{\alpha_j}]^\dagger = ((\alpha_{\underline{j}})_i E_{\alpha_{\underline{j}}})^\dagger$$

$$[E^\dagger_{\alpha_j}, H^\dagger_i] = (\alpha_{\underline{j}})^\dagger_i E^\dagger_{\alpha_{\underline{j}}}$$

$$[H_i, E^\dagger_{\alpha_j}] = -(\alpha_{\underline{j}})_i E^\dagger_{\alpha_{\underline{j}}}$$

using $H^\dagger = H$ (because the $H's$ are Hermitian) and $(\alpha_j)^\dagger_i = (\alpha_j)_i$ (because $(\alpha_j)_i$ is real). The result is that for every operator E_{α_j} in the non-Cartan sub-algebra, there is another operator $E_{-\alpha_j} = E^\dagger_{\alpha_j}$ in that same space. This pairing implies that $d-r$ is always even and j really only varies over $(d-r)/2$ as long as both $+\alpha$ *and* $-\alpha$ are included. The associated eigenvalues are α_j *and* $-\alpha_j$. We will now show that there are corresponding state vector equations that establish that there are associated eigenvectors or eigenstates.

In the adjoint representation, we have d $d\,x\,d$ matrices where d is the dimension of the Lie algebra i.e. the number of generators. With no loss of generality, we can define a set of d $1\,x\,d$

[8] Some authors refer to the root vectors as the roots but it seems clearer to have the term root refer to a scalar and the term root vector refer to the vector of scalar roots.

basic vectors as (1,0,0…), (0,1,00…), (0,0,1,0,0…) and so on. We can associate the first r of these basic vectors with the r $H_i\,'s$ and the last $d-r$ of these basic vectors with the $d-r\;E_{\alpha_j}\,'s$. The $H_i\,'s$ are all diagonal with the root values on the diagonal as shown below

$$H_i = \left(\begin{array}{ccc|ccc} 0_1 & & & & & \\ & \ddots & & & 0 & \\ & & 0_r & & & \\ \hline & & & \alpha_{1i} & & \\ & 0 & & & \ddots & \\ & & & & & \alpha_{(d-r)i} \end{array}\right) \tag{7.3.3}$$

Given this construction of the basic vectors and the H operators we have

$$H_i\left|H_j\right\rangle = 0 \;\; and \;\; H_i\left|E_{\alpha_j}\right\rangle = (\alpha_{\underline{j}})_i\left|E_{\alpha_{\underline{j}}}\right\rangle \tag{7.3.4}$$

which correspond to

$$[H_i, H_j] = 0 \;\; and \;\; [H_i, E_{\alpha_j}] = (\alpha_{\underline{j}})_i E_{\alpha_{\underline{j}}}$$

Also we require

$$\left\langle E_{\alpha_i}\right|\left|E_{\alpha_j}\right\rangle = \delta_{ij} \tag{7.3.5}$$

This construction permits us to refer to the $E_{\alpha_j}\,'s$ as operators or eigenstates whichever is most appropriate. Note that the Cartan subalgebra is an r dimensional subspace of the full Lie algebra. This subspace is spanned by r degenerate but independent eigenvectors with the same eigenvalue, 0. We state, without proof, that the root vectors, α_j, are unique so that the eigenvectors, $\left|E_{\alpha_j}\right\rangle$, are non-degenerate. This is why the eigenstates can be labeled with a root vector.

All of the developments in this section, so far, have been with the adjoint representation. The correspondence between states and operators is unique to the adjoint representation. If we were working in another representation, D, we would still find ‘eigenoperators’ and eigenvalues for the H_i operators, (in other words, equation (7.3.2) would still be correct) but equation (7.3.4) would need to be replaced by

$$H_i\left|\mu_j \;\; D\right\rangle = \left(\mu_{\underline{j}}\right)_i\left|\mu_{\underline{j}} \;\; D\right\rangle \tag{7.3.6}$$

Where D represents any non-adjoint representation. These eigenvalues are called weights and they are designated as $(\mu_j)_i$ and the vector μ_j is called a weight vector. $(\alpha_j)_i$, the roots, have the same values as the weights, $(\mu_j)_i$ in the adjoint representation. An alternative equation for (7.3.6) is

$$H_i\left|F_{\mu_j}\right\rangle = \left(\mu_{\underline{j}}\right)_i\left|F_{\mu_{\underline{j}}}\right\rangle. \tag{7.3.7}$$

with

$$\left\langle F_{\mu_i} \middle\| F_{\mu_j} \right\rangle = \delta_{ij} \tag{7.3.8}$$

The table below organizes the various notations used here for the operators and states

Table 7.3.1 – Notations Used for Adjoint and Non-Adjoint Representations

Representation		**Cartan Operators**	**Non-Cartan Operators**	**Non-Cartan States**
Name	**Symbol**			
Adjoint	A	H_i	E_{α_j}	$\left\|E_{\alpha_j}\right\rangle$
Non-Adjoint	D	H_i	E_{α_j}	$\left\|F_{\mu_j}\right\rangle$ or $\left\|\mu_j\ D\right\rangle$

Section 7.5 will discuss the underlying issues here further. The hope is that spending a little more time on these details will shed some light on subtleties that otherwise may cause confusion.

7.4. Commutation Relationships between the $E_{\alpha j}$

Up to now we have focused on the commutation relationships between the Cartan subalgebra operators, H_i, and the operators, E_{α_j}, in the complement of the Cartan subalgebra. We will need the commutation relationships between the E_{α_j}'s also.

Consider the commutator of H_i *and* $\left[E_{\alpha_j}, E_{\alpha_k}\right]$. Using the Jacobi Identity (see Section 8.1), we get

$$\left[H_i,\left[E_{\alpha_j}, E_{\alpha_k}\right]\right] = -\left[E_{\alpha_j},\underbrace{\left[E_{\alpha_k}, H_i\right]}_{=-\left(\alpha_{\underline{k}}\right)_i E_{\alpha_{\underline{k}}}}\right] - \left[E_{\alpha_k},\underbrace{\left[H_i, E_{\alpha_j}\right]}_{\left(\alpha_{\underline{j}}\right)_i E_{\alpha_{\underline{j}}}}\right]$$
$$= \left(\alpha_{\underline{k}}\right)_i \left[E_{\alpha_j}, E_{\alpha_{\underline{k}}}\right] - \left(\alpha_{\underline{j}}\right)_i \left[E_{\alpha_k}, E_{\alpha_{\underline{j}}}\right]$$

Switching indices in the second term on the RHS yields

$$= \left\{\left(\alpha_{\underline{j}}\right)_i + \left(\alpha_{\underline{k}}\right)_i\right\}\left[E_{\alpha_{\underline{j}}}, E_{\alpha_{\underline{k}}}\right]$$

Focusing on the case where $\left(\alpha_j\right)+\left(\alpha_k\right)=0$, it is clear that $\left[E_{\alpha_j}, E_{\alpha_k}\right]$ commutes with all the H_i's. This means that the commutator $\left[E_{\alpha_j}, E_{-\alpha_j}\right]$ is in the Cartan subalgebra and can be expressed as a linear combination of the H_i's. Thus

$$\left[E_{\alpha_j}, E_{-\alpha_j}\right] = k_i H_i$$

We use without proof that $k_i = \left(\alpha_j\right)_i$ i.e. the root vector, so that

$$\left[E_{\alpha_j}, E_{-\alpha_j}\right] = \left(\alpha_j\right)_i H_i \tag{7.4.1}$$

7.5. SU(2) as an Example of the Cartan Basis Analysis

At this point, it will useful to apply all the developments in sections 7.3 and 7.4 to the SU(2) group using first the adjoint representation and then the defined representation.

Adjoint Representation

First, we will work with the adjoint representation which is a 3d representation (because SU(2) has three generators) in equation (5.5.4) so

$$H_1 = X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The E_α's are linear combinations of X_1 *and* X_2 also in equation (5.5.4)such that they are raising and lower operators for H. Suitable linear combinations are $E_{\alpha_1} = (X_1 + iX_2)/\sqrt{2}$ *and* $E_{-\alpha_1} = (X_1 - iX_2)/\sqrt{2}$ so that

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; and \; E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We now change the basis so that H_1 takes the form of (7.3.3). We do this so that we can verify the operator equations and the state equations consistently. We use the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to apply a similarity transformation, SMS^{-1}, to obtain

$$H_1 = \left(\begin{array}{c:cc} 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right) \tag{7.5.1}$$

$$E_{\alpha_1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; and \; E_{-\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{7.5.2}$$

We could put primes on $H_1,\ E_{\alpha_1}\ and\ E_{-\alpha_1}$ but it doesn't seem necessary. Another comment on notation, we use the somewhat cumbersome notation of $H_1,\ \alpha_1\ and\ (\alpha_1)_1$ instead of just $H\ and\ \alpha$ so that when we consider SU(3) with a 2d Cartan subalgebra, we will use the identical notation and avoid confusion.

As an exercise, one can verify that equations (7.5.1) and (7.5.2) satisfy equation (7.3.2) with positive root, $(\alpha_1)_1$, equal to 1. That $(E_{\alpha_1})^\dagger = E_{-\alpha_1}$ is clear by inspection of equation (7.5.2) . And lastly, one can verify that equations (7.5.1) and (7.5.2) satisfy equation (7.4.1).

Having completed the operator view, we take a look at the state view, i.e. equation (7.3.4). Guided by the values on the diagonal of H_1, equation (7.5.1), we make the following assignments of state vectors to basis vectors

$$\left|H_1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \left|E_{\alpha_1}\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\ and\ \left|E_{-\alpha_1}\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

By inspection, equation (7.3.4) is seen to be satisfied, again with, $(\alpha_1)_1$, equal to 1.

To complete the use of SU(2) as an example of the Cartan basis analysis in the adjoint representation, we point out the that there are two root vectors, both 1d. The root vector for state E_{α_1} has the scalar value +1 and the root vector for state $E_{-\alpha_1}$ has the scalar value -1. Further on, we will plot root vectors in order to make geometrical observations. For SU(2), only a line is needed to make such a plot with two points at +1 and -1.

Defining Representation

Now we look at SU(2) from its 2d defining representation. From equation (5.5.2), the generators are

$$X_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},\quad X_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix},\quad X_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

so

$$H_1 = X_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

and

$$E_{\alpha_1} = (X_1 + iX_2)/\sqrt{2} = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

and

$$E_{-\alpha_j} = (X_1 - iX_2)/\sqrt{2} = \begin{pmatrix} 0 & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix}$$

These operators satisfy equations (7.3.2), $E_{-\alpha_j} = E_{\alpha_j}^{\dagger}$ and (7.4.1) with $(\alpha_1)_1 = 1$. Relative to the operator view there seems to be little different in form between the adjoint representation and the defining representation The difference is that the defining representation cannot support a state view that is consistent with the operator view because its dimensionality does not match the dimensionality of the state space of the Lie algebra. For SU(2), the state space of the Lie algebra is 3d but the defining representation space is only 2d. Therefore, states and eigenvalues still exist but the consistent machinery of operators and corresponding state vectors does not. The state eigenvalues are called weights and they have associated eigenvectors but the eigenvectors do not correspond to any operators the way root eigenvectors do. The governing equation is (7.3.7)

$$H_i \left| F_{\mu_j} \right\rangle = \left(\mu_{\underline{j}}\right)_i \left| F_{\mu_{\underline{j}}} \right\rangle.$$

The state view gives $(\mu_1)_1 = \frac{1}{2}$ and $(\mu_2)_1 = -\frac{1}{2}$ from the state equations below

$$H_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \textit{and} \ H_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1/2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $\left| F_{\mu_1} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\left| F_{\mu_2} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Equivalently, the weights can be taken directly from the diagonal elements of H_1. Notice that the weights do not appear in pairs as the roots do.

At this point, we have SU(2) with the weight vector for state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ being the scalar value +1/2 and the weight vector for state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ being the scalar value -1/2.

7.6. Observations on Roots, Weights, Operators and States

Having discussed the Cartan basis and seen how it works in both the adjoint and defined representations of SU(2), we can make some observations.

We make a distinction between the operator view and the state view. Equation (7.3.2),

$$[H_i, E_{\alpha_j}] = (\alpha_{\underline{j}})_i E_{\alpha_{\underline{j}}}, \tag{7.6.1}$$

is noted as an operator equation. And equation (7.3.7),

$$H_i \left| F_{\mu_j} \right\rangle = \left(\mu_{\underline{j}}\right)_i \left| F_{\mu_{\underline{j}}} \right\rangle, \tag{7.6.2}$$

is noted as a state equation,

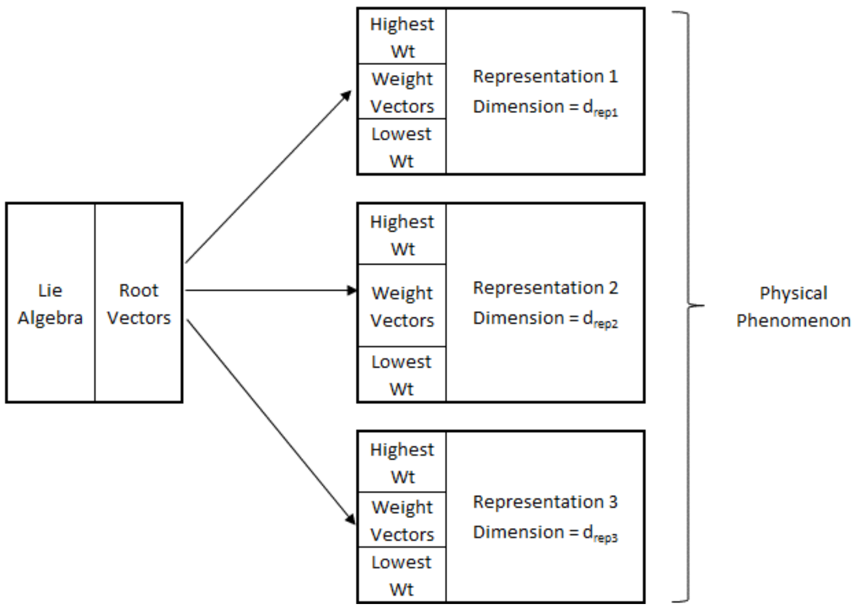
An operator equation is determined by the Lie algebra because it applies to the generators directly, independently from any particular representation of those generators. As a result, the α_j's are also independent of any particular representation. This is reflected in the dimensionality of the α_j's namely, there are $d-r$ root vectors and each one is a $1\ x\ r$ vector and these values do not change with representation.

The state equation, on the other hand, is determined by the representation (as well as the Lie algebra) because the state vectors exist in the representation space. Thus, the μ_j's for different representations are different. This dependency on representation is reflected by the fact that the number of weight vectors is the dimension of the representation, d_{rep}, although the dimensionality of each weight vector is $1\ x\ r$, the same as the root vectors.

One result of the above is that any representation (other than the adjoint) yields both the roots and weights. The roots are obtained from the operator equation and the weights are obtained from the state equation. Equivalently to using the state equation, the weights can be obtained from the diagonals of the H's. This will all be demonstrated in the next section using SU(3).

It is often said that the roots are the weights of the adjoint representation. This is numerically true, however, it is misleading because it sounds as if roots and weights are the same kind of thing. But the nature of roots and weights are different. First, as we have shown, the roots are determined by the Lie algebra whereas the weights are determined by the representation (and the Lie algebra). Second, roots raise and lower weights but weights do not raise and lower roots. Third, weights, determined by the representations, actually have physical manifestations whereas the roots, determined by the Lie algebra, act only indirectly through the weights. It would be more accurate, then, to say that in the adjoint representation the roots are *equal* to the weights rather than the roots *are* the weights. Figure 7.6.1 below illustrates these points.

Figure 7.6.1 – The Context of Roots and Weights



For completeness, we point out that there are two other equations, one an operator equation, (7.4.1)

$$\left[E_{\alpha_j}, E_{-\alpha_j}\right] = \left(\alpha_j\right)_i H_i, \tag{7.6.3}$$

and the other a state equation, (7.3.8)

$$\left\langle F_{\mu_i} \middle\| F_{\mu_j} \right\rangle = \delta_{ij}, \tag{7.6.4}$$

that do not enter into these considerations because they are only providing a normalization criteria for the operator E_{α_j} and the state $\left| F_{\mu_j} \right\rangle$. In other words, if equation (7.6.1) is satisfied by an E_{α_j}, it will also be satisfied by $k\,E_{\alpha_j}$. Equation (7.6.3) determines k. And similarly, if equation (7.6.2) is satisfied by an F_{μ_j}, it will also be satisfied by $l\,F_{\mu_j}$. Equation (7.6.4) determines l.

7.7. SU(3) as an Example of the Cartan Basis Analysis

SU(3) will provide a richer example for the Cartan basis analysis because it's Cartan subalgebra is 2d so that the root and weight vectors are no longer scalars as they were with SU(2).

<u>Defining Representation</u>

We start with the eight generators of SU(3) as 1/2 of the Gell-Mann matrices:

$$X_1 = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & -i/2 & 0 \\ i/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} \quad X_5 = \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & 0 & 0 \\ i/2 & 0 & 0 \end{pmatrix}$$

$$X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \quad X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i/2 \\ 0 & i/2 & 0 \end{pmatrix} \quad X_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Using X_3 *and* X_8 as our H operators since they are already diagonal assuring that they commute, we get:

$$H_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; and \; H_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We create paired raising and lowering operators with the other generators

$$E_{\pm\alpha_1} = (X_1 \pm iX_2)/\sqrt{2}$$
$$E_{\pm\alpha_2} = (X_4 \pm iX_5)/\sqrt{2}$$
$$E_{\pm\alpha_3} = (X_6 \pm iX_7)/\sqrt{2}$$

Yielding the raising operators

$$E_{\alpha_1} = \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_{\alpha_2} = \begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_{\alpha_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}$$

with the lowering operators, $E_{-\alpha_i}$ *with* $i = 1$ *to* 3, being the transposes since they are all real.

We extract the weights from the diagonal elements of H_1 *and* H_2 (or use equation (7.6.2) $H_i \left| F_{\mu_j} \right\rangle = \left(\mu_{\underline{j}}\right)_i \left| F_{\mu_{\underline{j}}} \right\rangle$.) giving

$$\mu_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \quad \mu_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \quad \mu_3 = \left(0, -\frac{1}{\sqrt{3}}\right) \tag{7.7.1}$$

and we use equation (7.3.2)

$$[H_i, E_{\alpha_j}] = (\alpha_{\underline{j}})_i E_{\alpha_{\underline{j}}} \tag{7.7.2}$$

to calculate the roots obtaining

$$\alpha_1 = (1, 0) \quad \alpha_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad \alpha_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \tag{7.7.3}$$

Adjoint Representation
We do not exert the effort to actually find the adjoint representation (as it is eight dimensional!) since we already have both the weights and the roots.

7.8. How the Roots Act on the Weights

It is often said that the roots are the differences between weights. We prefer to look at this phenomenon as one weight is another weight +/- a root which corresponds to the raising and lowering operations. In figure 7.8.1, we show this for SU(3).

The figure shows the 2d root and weight vectors plotted with the eigenvalues for H_1 on the horizontal axis and those for H_2 on the vertical axis. The open circles show the six roots (three positive and the corresponding three negative). The black circles show the three weights. The red circles show the result of adding all six roots to the first weight, μ_1, where the addition of the positive roots is shown as solid arrows and the addition of the negative roots is shown as dashed arrows.

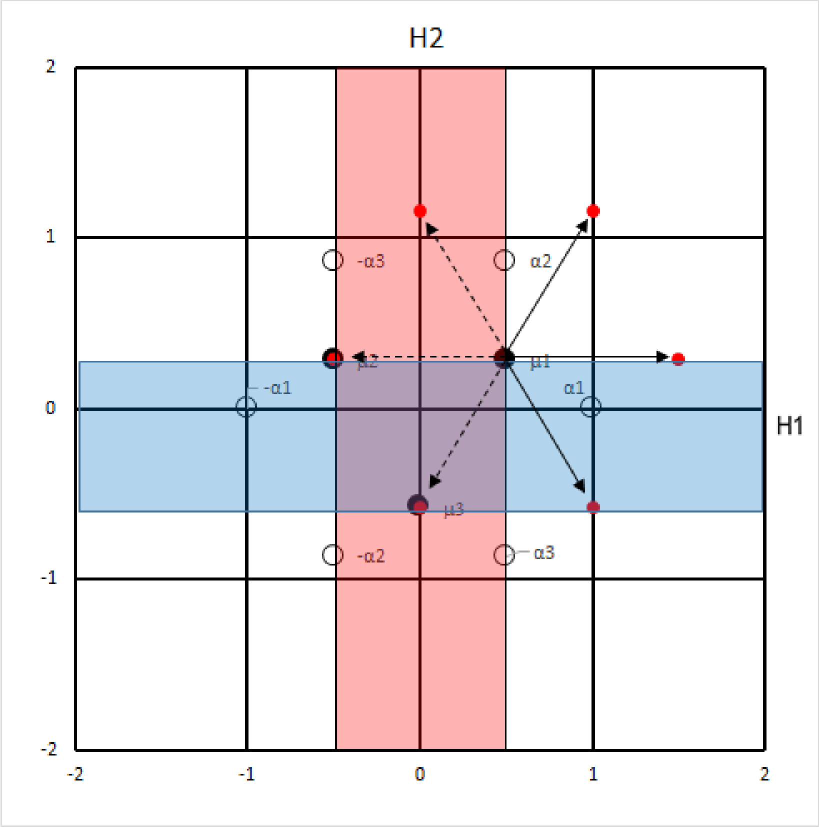
The red area shows the region of all weights for which μ_1 is the highest weight and μ_2 is the lowest weight. In other words,

$$\mu_1 - \mu_{red} \ \ is \ \ positive \ \ and$$
$$\mu_{red} - \mu_2 \ \ is \ \ positive$$

The red area shows the limits of permissible weights when we consider H_1 first and H_2 second but that order is arbitrary. If we reverse the order, then $\mu_1,\ now\left(\frac{1}{2\sqrt{3}},\frac{1}{2}\right)$, is still the highest weight but $\mu_3,\ now\left(-\frac{1}{\sqrt{3}},0\right)$, becomes the lowest weight. The blue area then shows the limits of permissible weights from this point of view. The intersection of the red and blue areas shows the actual limits of permissible weights.

Of the six roots, two lead from μ_1 to another weight, i.e. $-\alpha_1 \ \ and \ \ -\alpha_2$ lead to $\mu_2 \ \ and \ \ \mu_3$ respectively. $-\alpha_3$ and all three positive roots lead to values which cannot be weights.

Figure 7.8.1 – Roots and Weights of SU(3) and μ_1 Raised and Lowered



The constraints on weights are relatively straightforward. There are also constraints on the roots, but these are deep in the Lie algebra and are discussed in the next section.

7.9. Constraints on the Roots

The group condition combined with the continuous, smooth parameters of a Lie group impose strong constraints on the roots. We will not derive these results here but just show them.

The first is expressed by

$$\frac{2\alpha_i \cdot \mu_j}{\alpha_i^2} \text{ is an integer} \tag{7.8.1}$$

The table below, (7.8.4), shows the 9 integer values of this expression for SU(3) from the weights, (7.7.1),

$$\mu_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \quad \mu_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \quad \mu_3 = \left(0, -\frac{1}{\sqrt{3}}\right) \tag{7.8.2}$$

and the roots, (7.7.3),

$$\alpha_1 = (1, 0) \quad \alpha_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad \alpha_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \tag{7.8.3}$$

that we previously calculated.

$$\boxed{\frac{2\alpha_i \cdot \mu_j}{(\alpha_i)^2}} \begin{pmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 & \boldsymbol{\mu}_3 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ -\boldsymbol{\alpha}_3 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{7.8.4}$$

Note that $-\alpha_3$ was used so that only positive roots were used in the table. This table just demonstrates that SU(3) meets the constraint.

The second expression of how the roots are constrained explains why the expression in (7.8.1) must be an integer.

$$\frac{2\alpha_i \cdot \mu_j}{\alpha_i^2} = -\left(p_{ji} - q_{ji}\right) \tag{7.8.5}$$

where p_{ji} is a non-negative integer, the maximum times that state μ_j can be raised by α_i and q_{ji} is a non-negative integer, the maximum times that state μ_j can be lowered by α_i. Georgi calls this the master equation because it is so important.

Figure 7.8.1 can be used to determine the values of p_{ji} *and* q_{ji}

$$\boxed{p} \quad (\boldsymbol{\mu}_1\ \boldsymbol{\mu}_2\ \boldsymbol{\mu}_3) \qquad \boxed{q} \quad (\boldsymbol{\mu}_1\ \boldsymbol{\mu}_2\ \boldsymbol{\mu}_3) \quad \boxed{-(p-q)}(\boldsymbol{\mu}_1\ \ \boldsymbol{\mu}_2\ \ \boldsymbol{\mu}_3)$$

$$\begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \textbf{-}\boldsymbol{\alpha}_3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \textbf{-}\boldsymbol{\alpha}_3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \textbf{-}\boldsymbol{\alpha}_3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

And we note that the $-(p-q)$ matrix is the same as (7.8.4), as expected.

The constraints so far have involved both roots and weights. What we would like is to formulate constraints that only involve roots because these constraints are determined by the Lie algebra and will be true for all representations. We can apply (7.8.5) to two roots, for example α_1 *and* α_2, if we restrict ourselves to the adjoint representation. Then $\alpha_i = \alpha_1$ *and* $\mu_j = \alpha_2$ giving

$$\frac{2\alpha_1 \cdot \alpha_2}{\alpha_1^{\ 2}} = -\left(p_{21} - q_{21}\right)$$

We can do this because in the adjoint representation the weights are numerically equal to the roots. Then, we let $\alpha_i = \alpha_2$ *and* $\mu_j = \alpha_1$ giving

$$\frac{2\alpha_2 \cdot \alpha_1}{\alpha_2^{\ 2}} = -\left(p_{12} - q_{12}\right)$$

Multiplying LHSs and RHSs and dividing both sides by four gives

$$\frac{\left(\alpha_1 \cdot \alpha_2\right)^2}{\alpha_1^2\,\alpha_2^2} = \left(\frac{\alpha_1 \cdot \alpha_2}{|\alpha_1||\alpha_2|}\right)^2 = \cos^2(\theta_{\alpha_1\alpha_2}) = \frac{1}{4}\left(p_{21} - q_{21}\right)\left(p_{12} - q_{12}\right) \qquad (7.8.6)$$

Where $\theta_{\alpha_1\alpha_2}$ is the angle between root vector α_1 and root vector α_2. Since $\left(p_{12} - q_{12}\right)\left(p_{21} - q_{21}\right)$ must be a non-negative integer (non-negative since $\cos^2$ is non-negative), we only have 5 possible values $0, 1, 2, 3, 4$ because $\cos^2 \le 1$ These five values correspond to θ values of 90^0, 60^0 *or* 120^0, 45^0 *or* 135^0, 30^0 *or* 150^0, *and* 0^0 *or* 180^0 respectively. Thus the angles between roots are very tightly constrained.

We can compare these results with figure 7.8.1. The roots form a hexagon so adjacent roots are 60^0 apart and roots separated by 1 root are 120^0 apart as allowed by equation (7.8.6). We note that 0^0 *and* 180^0 are not of interest since 0^0 is eliminated because the roots are all unique, i.e. the multiplicity of states of non-zero roots is 1 and 180^0 is a given by the fact that all roots come in pairs, one the negative of the other.

These angles provide a powerful tool for categorizing Lie algebras that is beyond the scope of this review.

7.10. Simple Roots

A subset of the roots, called the simple roots, is the most compact descriptor of the Lie algebra. The simple roots are any positive root that cannot be expressed as the sum of two other positive roots. For SU(3), there are two simple roots

$$\alpha_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad \alpha_3 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

The other positive root, $\alpha_1 = (1, 0)$, is the sum of the two simple roots.

The number of simple roots is equal to the rank of the Lie algebra. With the simple roots the entire Lie algebra can be reconstructed. Possibly most importantly, all of the irreducible representations of the Lie algebra can be reconstructed. We leave how this is done to other texts.

7.11. Summary of Main Points

The adjoint representation is the only representation where there is a correspondence between the operator view of the Cartan basis and the state view. In the defined representation, these two views are not consistent because the defined representation does not have the correct dimensionality to support a proper state view.

Roots appear in the adjoint representation both as the diagonal element of the H matrices (equivalent to the eigenvalues of the state view) and as eigenvalues in the operator view.

Weights appear in the defined representation as the diagonal elements of the H matrices (equivalent to the eigenvalues of the state view) and roots appear as eigenvalues in the operator view.

SU(2) provides an instructive example because the dimensionality of the adjoint representation is different from the dimensionality of the defined representation. SU(3) provides an instructive example because the rank of the Cartan algebra is 2 so that weight vectors and root vectors are non-trivial.

A visual presentation of how root vectors operate on the weight vectors includes a description of constraints on the weights. The much more subtle constraints that exist on the roots is then discussed. Lastly, simple roots, the most concise description of the Lie algebra, completes the discussion.

8. Additional Material

8.1. Lie Algebras

This section discusses some interesting aspects of Lie algebras that are not needed for the rest of this review.

A Lie algebra is a vector space, L, together with any bilinear operation satisfying

$$\begin{aligned}
[a+b,\ c] &= [a,\ c]+[b,\ c] \\
[ka,\ b] &= k[a,\ b] \\
[a,\ b] &= -[b,\ a] \\
0 &= [a,\ [b,\ c]]+[b,\ [c,\ a]]+[c,\ [a,\ b]] \quad \textit{Jacobi Identity} \\
&\textit{where } a,\ b,\ c \in L \textit{ and } k \in R \textit{ or } C.
\end{aligned}$$

The bilinear operation $[,]$ is called a Lie bracket. The commutator $[a,\ b]=ab-ba$ satisfies the requirements of a Lie bracket and, in fact, is the usual operation for physics applications.

A Lie group is a manifold where each point is a group element. The Lie algebra is the tangent space at the identity element. The generators span that tangent space.

A typical manifold may have a metric defined on it which relates two separate points on the manifold by specifying a distance between the two points. But a manifold supporting a Lie algebra maintains a more intimate relationship between two points on the manifold. Since each point is a group element, any ordered pair of points, e.g. g1 and g2, leads to a third point that must also be on the manifold corresponding to g3 = g2 g1. This group condition puts severe constraints on parameters of the Lie algebra, e.g. the root vectors.

8.2. SU(N), SO(2N) and SO(2N+1)

Table 8.2.1. Information on the Lie Algebras of SU(N), SO(2N) and SO(2N+1)

				Defining Rep.		**Adjoint** Rep.		
Group		# of Generators (d)	Rank (r)	Dim.	# of $\vec{Wts}$	Dim.	# of $+\vec{Rts}$	# of Simple $\vec{Rts}$
SU(N)	**N**	**$N^2 - 1$**	**N-1**	**N**	**N**	**d**	**N(N - 1)**	**r**
SU(2)	2	3	1	2	2	3	1	1
SU(3)	3	8	2	3	3	8	3	2
SU(4)	4	15	3	4	4	15	6	3
SU(5)	5	24	4	5	5	24	10	4
...								
SU(10)	10	99	9	10	10	99	45	9
SO(2N)	**N**	**N(2N - 1)**	**N**	**2N**	**r**	**d**	**N(N - 1)**	**r**
SO(2)	1	1	1	2	1	1	0	1
SO(4)	2	6	2	4	2	6	2	2
SO(6)	3	15	3	6	3	15	6	3
SO(2N+1)	**N**	**N(2N+1)**	**N**	**2N + 1**	**r**	**d**	**r^2**	**r**
SO(3)	1	3	1	3	1	3	1	1
SO(5)	2	10	2	5	2	10	4	2
SO(7)	3	21	3	7	3	21	9	3

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